Engineering Dynamics Reference

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Contents

Part I

Dynamics of Particles

Chapter 1

Kinematics

Kinematics refers to the way in which we express (usually mathematically) how motion occurs. It does not attempt to attribute a *cause* for the motion. Instead, it is our way of describing motion that is consistent with definitions of position, velocity, and acceleration, along with constraints imposed on a mechanical system.

1.1 Basic Elements

The basic elements of kinematics are position, velocity, and acceleration. We define them below.

1.1.1 Position

Suppose we have a particle located in space as indicated in the figure below, and suppose we have chosen an *origin* in that space. **Position**, $\vec{\boldsymbol{r}}$, is a *vector* that extends from the origin to the location of the particle.

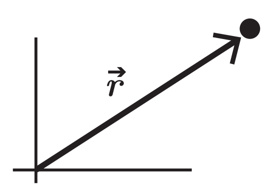

Figure 1.1: Position vector.

Generally, in dynamics, the particle moves. Thus, we think of position as a function of time: $\vec{r}(t)$.

Physical dimensions: The quantities we used to describe physical systems are comprised of base quantities that we call "physical dimensions." Physical dimensions in mechanics are comprised in terms of mass (M), length (L), and time (T). The physical dimensions of position are length, L.

1.1.2 Velocity

Velocity, $\vec{v}$ is the derivative of position with respect to time:

$$\vec{v}(t_0) = \left.\frac{d\vec{r}}{dt}\right|_{t_0} = \lim_{\Delta t \to 0} \frac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t}. \tag{1.1}$$

It is important to note that this is a vector derivative. Velocity, itself, is a vector. The magnitude of the velocity vector is often called the **speed**.

Direction of the Velocity Vector. Based on the definition of the velocity vector (1.1), one can determine that the direction of the velocity vector has two primary properties.

1. As an object moves along a smooth path, *the velocity vector is tangent to the path.*

2. The *velocity vector points in the direction of motion.*

To see why the velocity vector has these properties, you may click on the links to the following videos:

- *Derivative of the Position Vector: Motion Along a Straight Line.*

- *Velocity Vector as the Derivative of Position: Motion on a Curved Path.*

Physical dimensions: Since position is a length and our definition of velocity has a division by time, the physical dimensions of velocity are: L/T.

1.1.3 Acceleration

Acceleration, $\vec{a}$ is the derivative of the velocity vector with respect to time:

$$\vec{\boldsymbol{a}}(t_0) = \left.\frac{d\vec{\boldsymbol{v}}}{dt}\right|_{t_0} = \lim_{\Delta t \to 0} \frac{\vec{\boldsymbol{v}}(t_0 + \Delta t) - \vec{\boldsymbol{v}}(t_0)}{\Delta t} \tag{1.2}$$

Physical dimensions: Since acceleration is a rate of change of velocity with respect to time, the physical dimensions are: L/T^2.

1.2 Motion Along a Straight Line

When an object moves along a straight line, its acceleration is particularly simple. The reader is encouraged to watch the video at this link (*Derivative of the Velocity Vector: Motion Along a Straight Line*) to see how the velocity is constructed from (1.2) in this special circumstance. The main take-aways are:

1. The acceleration vector is tangent to the line of motion.
2. When the acceleration vector and velocity vector are in the same direction, the object speeds up.
3. When the acceleration vector and velocity vector are in opposite directions, the object slows down.

1.3 Path Coordinates

Suppose an object is moving from left to right along the curved path shown below. Location along the path can be parameterized by the variable s which denotes a distance traveled along the path.

1.3.1 Basis Vectors

Path coordinates is a means by which to express important quantities such as velocity and acceleration. To do so, we use basis vectors $\hat{\boldsymbol{e}}_{\boldsymbol{t}}$ and $\hat{\boldsymbol{e}}_{\boldsymbol{n}}$. The vector $\hat{\boldsymbol{e}}_{\boldsymbol{t}}$ is always *tangent to the path*, pointing in the direction of travel.

The vector $\hat{\boldsymbol{e}}_{\boldsymbol{n}}$ is *perpendicular to the path*, pointing *inward* toward the center of curvature. At inflection points and segments where the path is straight, $\hat{\boldsymbol{e}}_{\boldsymbol{n}}$ is not uniquely defined.

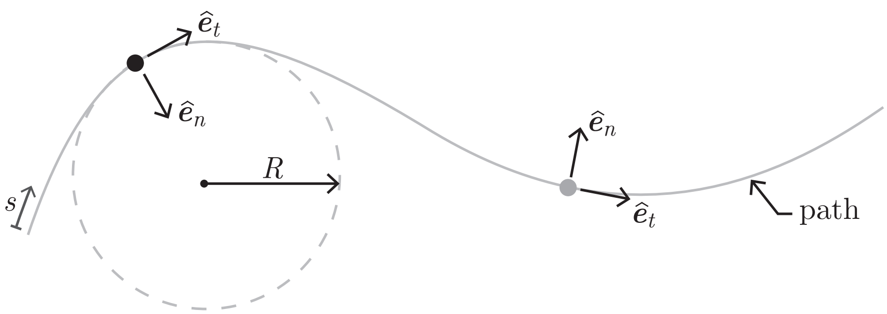

Figure 1.2: Motion along a curved path.

1.3.2 Velocity in Path Coordinates

The velocity vector is always tangent to the path, and points in the direction of motion. Thus we can write it as

$$\vec{\boldsymbol{v}} = v\,\hat{\boldsymbol{e}}_{\boldsymbol{t}}. \tag{1.3}$$

The component v is the speed of the object. And since the speed can be written as $v = \dot{s}$, we can write the velocity as $\vec{\boldsymbol{v}} = \dot{s}\,\hat{\boldsymbol{e}}_{\boldsymbol{t}}$.

1.3.3 Acceleration in Path Coordinates

Recall that when an object moves along a straight line, acceleration is always tangent to that line of motion. However, when the path is curved, the acceleration is generally *not* tangent to the path. The acceleration normally has component tangent and perpendicular to the path and can be written as

$$\vec{\boldsymbol{a}} = \dot{v}\,\hat{\boldsymbol{e}}_t + \frac{v^2}{R}\,\hat{\boldsymbol{e}}_n. \tag{1.4}$$

The first term in (1.4) occurs if the object in Figure 1.2 is speeding up or slowing down. It happens when the *magnitude* of the velocity vector changes. This component of acceleration behaves similar to acceleration in straight line motion as described in Section 1.2.

The second term in (1.4) is called the *centripetal acceleration.* It occurs *because the velocity vector is changing its direction.* This component of velocity always points inward. If you double the speed of the object, its centripetal acceleration quadruples.

The denominator contains the *radius of curvature*, R. Imagine drawing a circle that is tangent to the path at the location of the object (matching first derivative). Furthermore, suppose that the curvature of the circle matches the curvature of the path (second derivative). The radius of this circle, shown in Figure 1.2 is the radius of curvature.

If you can express the path as a function $y(x)$, then the radius of curvature can calculated by

$$R = \left| \frac{(1+y')^{\frac{3}{2}}}{y''} \right|, \qquad \text{where} \quad y' = \frac{dy}{dx}, \quad \text{and} \quad y'' = \frac{d^2y}{dx^2}. \tag{1.5}$$

1.4 Polar Coordinates

More to come.

Chapter 2

Force and Quantities Derived from Force

Definition: A force is an *interaction between objects* that provides a *push or a pull* between the objects. It has magnitude and direction and is therefore a *vector*.

Physical dimensions: In terms of mass (M), length (L), and time (T), the physical dimensions for force are $\frac{ML}{T^2}$. If you forget this, it's easiest to recall it via Newton's Second Law, $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$.

Units of measure: Newton (N), pound (lb).

2.1 Some Common Forces

2.1.1 Gravity

Any two objects with mass experience an attractive gravitational force. However, when we mechanical engineers talk about gravity, we're usually referring to the force of Earth pulling on other objects. The figure below is obviously not drawn to scale.

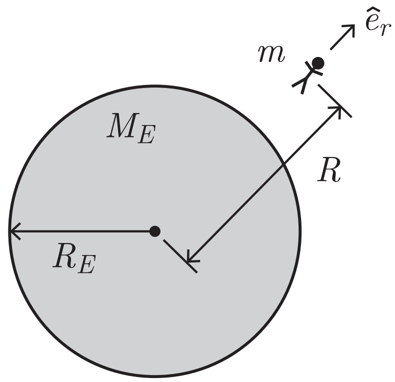

Figure 2.1: Earth's gravity pulling on an object.

Newton's inverse square model for gravity is

$$\vec{\boldsymbol{F}} = -m \frac{GM_E}{R^2} \hat{\boldsymbol{e}}_r. \tag{2.1}$$

Here, M_E and m are the masses[1] of the Earth and of the object respectively. R is the distance of the object from the center of the Earth. Finally, G is the Gravitational Constant, determined experimentally to be

$$G = 6.674 \times 10^{-11} \text{m}^3 \, \text{kg}^{-1} \, \text{s}^{-2}.$$

The term GM_E/R^2 in (2.1) is the gravitational field of Earth. It tells you how hard the earth pulls on the object *per unit mass* of the object. If we consider objects at or near the surface of Earth and set $R = R_E$, we get

$$\vec{\boldsymbol{F}} \approx -m \frac{GM_E}{R_E^2} \hat{\boldsymbol{e}}_r = -mg \, \hat{\boldsymbol{e}}_r. \tag{2.2}$$

This lower case g is the field strength of gravity on Earth. Its value is approximately 9.81 N/kg or 32.2 lb/slug. Note that a Newton per kilogram (N/kg) is equivalent to a meter per second squared (m/s^2) and a pound per slug (lb/slug) is equivalent to a foot per second squared (ft/s^2).

[1] Technically, physicists refer to these as *gravitational* masses to distinguish them from *inertial* masses. Since we are engineers interested in practical applications and physicists have not been able to measure a difference between gravitational and inertial masses, we do not make the distinction.

Physical dimensions: Since physical dimensions in mechanics are expressed in terms of mass (M), length (L), and time (T), the physical dimensions of g are L/T^2.

Also, some people (I'm not one of them) write $\vec{\boldsymbol{g}}$ as a vector:

$$\vec{\boldsymbol{F}} \approx m\,\vec{\boldsymbol{g}}, \qquad \text{where} \quad \vec{\boldsymbol{g}} = -\frac{GM_E}{R_E^2}\,\hat{\boldsymbol{e}}_r. \tag{2.3}$$

2.1.2 Springs

Springs, like everything else, have mass. However, in dynamic analysis, the inertia of springs are typically neglected and their role is to provide forces. When a spring has no axial force acting on it, it has natural length of L_o as depicted in Figure 2.2.

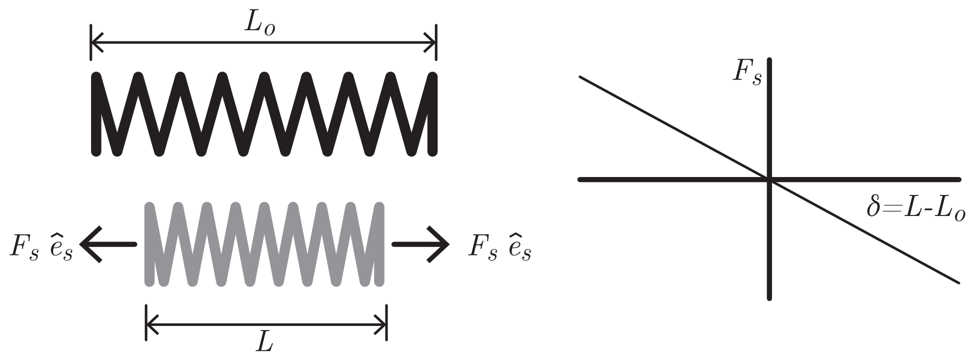

Figure 2.2: Spring.

When the spring is compressed so that its current length, L, is less than its natural length, it pushes outward with a force $F_s\,\hat{\boldsymbol{e}}_s$ as shown in the figure. When stretched, $L > L_o$, the spring pulls inward.

Under most normal circumstances, the spring force has a linear relationship with spring deformation:

$$F_s = -k(L - L_o). \tag{2.4}$$

The coefficient, k, is called the *spring constant* or *spring stiffness*. It determines the slope of the force/deformation relationship shown on the right side of Figrue 2.2.

Physical dimensions: Since k is a relationship between spring *force* and a change in spring *length*, it must have physical dimensions of M/T^2.

Units of measure: Typically Newton per centimeter (N/cm) or pounds per inch (lb/in).

2.1.3 Contact Forces

The left side of Figure 2.3 shows two surfaces in contact. The right side shows the forces of contact between the two object. As is customary, we

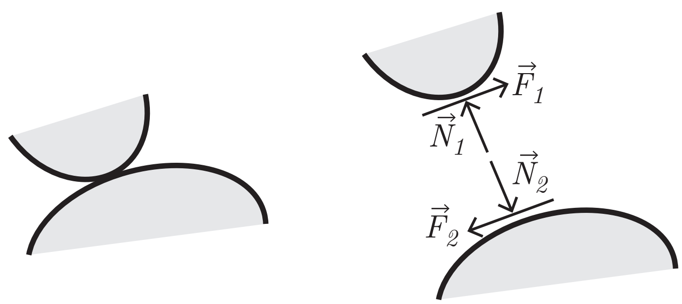

Figure 2.3: Surfaces in contact and the forces of contact.

split the forces into components perpendicular to the surface (normal force) and tangent to the surface (friction force). Furthermore, Newton's third law tells us that $\vec{\boldsymbol{N}}_1 = -\vec{\boldsymbol{N}}_2$ and $\vec{\boldsymbol{F}}_1 = -\vec{\boldsymbol{F}}_2$.

Normal Force

The purpose of the normal force is to prevent the bodies from penetrating each other. In dynamics problems, one typically solve for the normal force by solving this constraint.

Coulomb Friction Model

Friction is an immensely complicated phenomenon. However, there is a reasonably good empirical[2] model of friction that is suitable in many applications.

[2] An "empirical" model is one based on physical experiments and observations rather than some underlying theory.

According to the model, if the two surfaces are *sliding relative to each other*, then the friction is **kinetic friction** with magnitude

$$|\vec{\boldsymbol{F}}| = \mu_k \, |\vec{\boldsymbol{N}}|. \tag{2.5}$$

According to the model, friction only depends on the magnitude of the normal force, not other quantities such as speed. The direction of the kinetic friction vector is tangent to the surfaces of contact and it opposes the relative sliding motion.

Static friction is the type of friction that prevents sliding. It is important to note that "sliding" refers to a skidding, scraping, or scrubbing type of motion. It is possible for surfaces to roll on each other without slipping.

Calculating a static friction force is a matter of calculating the force necessary to prevent sliding. However, static friction has its limits. Eventually, if you push hard enough, surfaces will usually start to slide. According to the Coulomb model, the limit is given by

$$|\vec{\boldsymbol{F}}| \leq \mu_s |\vec{\boldsymbol{N}}|. \tag{2.6}$$

The constants μ_k and μ_s are the **coefficient of kinetic friction** and **coefficient of static friction**, respectively. Values of these coefficient depend on the nature of the surfaces in contact. As you might guess the friction coefficients for rubber on pavement are different from those of steel on ice.

Regardless of the nature of the materials in contact, though, the friction coefficients must satisfy the following inequality:

$$\mu_k \leq \mu_s. \tag{2.7}$$

To understand why this inequality is important, we must first discuss some implementation issues.

Implementation of the Coulomb Friction Model. When analyzing a problem with friction, you might need to determine if friction is kinetic or static, and when the nature of friction in a model switches from kinematic to static or from static to kinetic. In solving dynamics problems, the way that the Coulomb friction model is implemented is summarized in Figure 2.4.

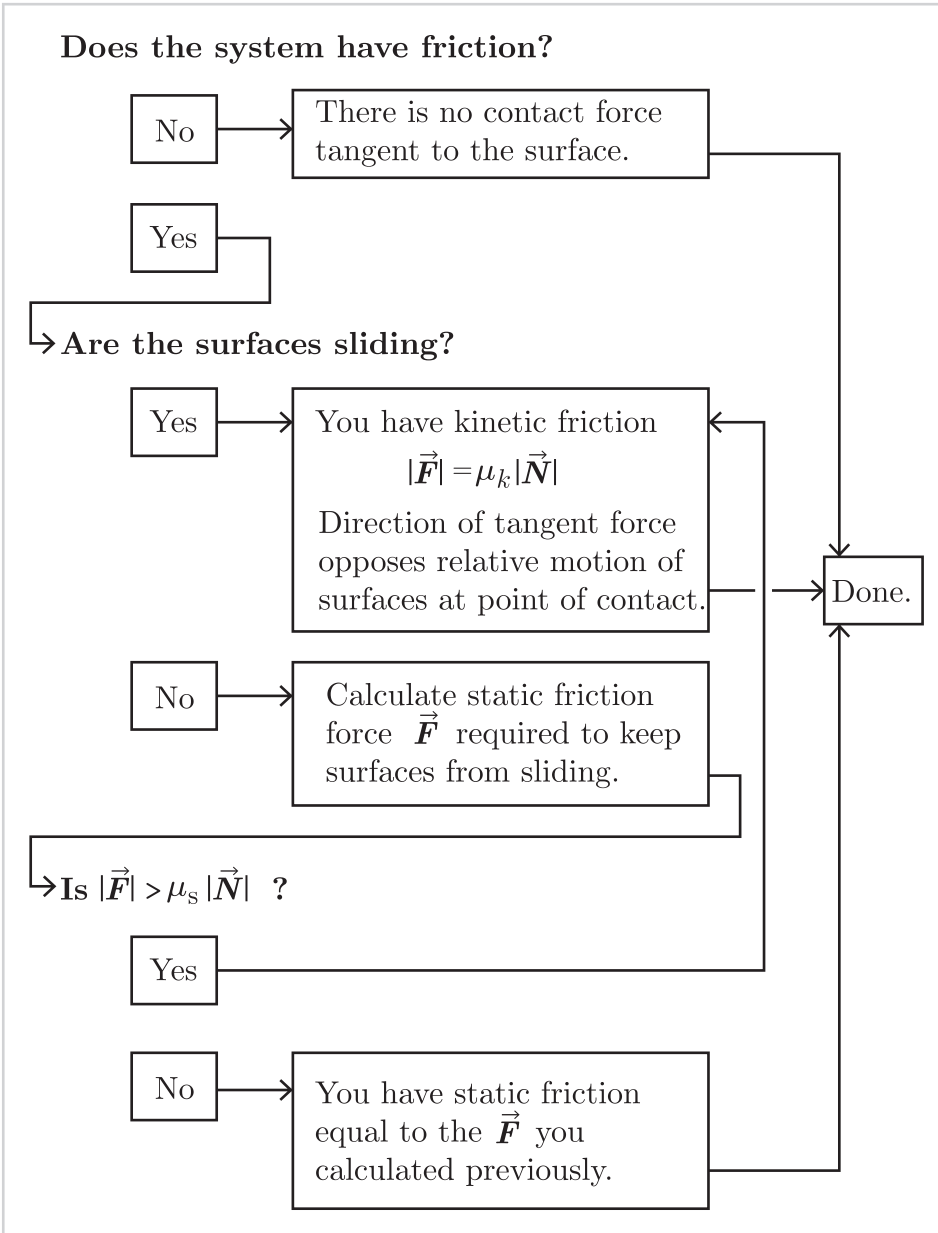

Figure 2.4: Implementation of the Coulomb friction model.

To understand why the inequality (2.7) is critical, consider the following thought experiment.

1. Suppose a heavy cardboard box resting on a table top as shown in Figure 2.5.
2. Apply a small horizontal push, $\vec{P}$, to the box. The static friction force, $\vec{F}$, will exactly balance out the push, and prevent the box from sliding.
3. Slowly increase the magnitude of the horizontal push force $\vec{P}$. As long as $|\vec{F}| < \mu_s|\vec{N}|$, the box will remain stuck.
4. As soon as the magnitude of $\vec{P}$ exceeds $\mu_s|\vec{N}|$, static friction cannot balance the push, so the box begins sliding.
5. While the magnitude of the push force, $|\vec{P}|$, slightly exceeds $\mu_s|\vec{N}|$, the friction force is that of *kinetic* friction and has magnitude $|\vec{F}| = \mu_k|\vec{N}|$.
6. Since, according to (2.7), μ_k is less than μ_s, the magnitude of the friction force is smaller than that of the push. Therefore the box begins accelerating to the right.

Figure 2.5: Box with friction on a tabletop.

But what would happen if the kinetic coefficient, μ_k was larger than the static coefficient, μ_s, contrary to (2.7)? In this case, the kinetic friction force in state 6 above would have magnitude larger than that of the push force $\vec{P}$. The kinetic friction force would rapidly bring the box to a stop again, returning to state 4 above. But in state 4, the box breaks free...

So if (2.7) were *not* true, then the box would enter a continuous stick, slip, stick, slip, stick, slip, ... loop. This contradicts our common experience.

Physical dimensions: Examination of Equations (2.5) and (2.6), reveals that the coefficients of frictions are each ratios of forces. Therefore, these quantities are dimensionless.

2.2 Work

The word "work" has several common meanings in the English language. In mechanics, work is defined as a path integral:

$$U_{A\to B} = \int_A^B \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}. \tag{2.8}$$

It is this definition of work that is used in the *Work-Energy Principle* (Section 3.3) and provides new techniques for analysis.

To be more specific, we should call $U_{A\to B}$ the work perform by force $\vec{\boldsymbol{F}}$ on an object as it moves from point A to point B along a specified path. Figure 2.6 provides an illustration in which the path has a "loop" in it.

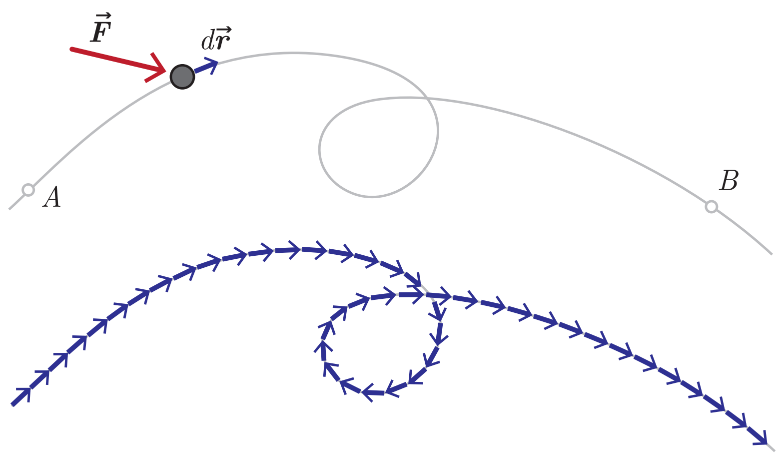

Figure 2.6: Work is a path integral.

The vectors $d\vec{\boldsymbol{r}}$ represent small steps along the path, just like the dx represents small steps along the number line in the type of integral you're probably more familiar with: $\int f(x)\,dx$. The bottom part of the figure shows a collection of $d\vec{\boldsymbol{r}}$ vectors that define the path.

Therefore, we can think of work as a summation:

$$U_{A\to B} \approx \sum \left(\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}\right). \tag{2.9}$$

For each vector $d\vec{\boldsymbol{r}}$ along the path, we are taking the dot product with the corresponding force vector $\vec{\boldsymbol{F}}$ at that point on the path, and adding the results. Each dot product is a scalar (i.e. just a number, so the summation in (2.9) produces a scalar. In the limit as the $d\vec{\boldsymbol{r}}$ vectors get smaller and smaller, the vectors become tangent to the path and the summation in (2.9) approaches the work integral in (2.8).

Observations: Here are some important properties of work that come directly from the definition (2.8).

1. *Work is a scalar, either positive, negative, or zero.* Work is the result of a **force** acting on an object as it moves. But because of the dot product in the definition, work is a scalar. It is just a number. Since the dot product may be positive, negative, or zero, the work may be positive, negative or zero.

2. *Work generally depends on path.* Figure 2.6 shows a specific path that the object follows as it travels from point A to point B. If the object took a different path in going from A to B, the $d\vec{\boldsymbol{r}}$ vectors would be different. Thus the integral (2.8) would be different. Therefore, in calculating the work in going from point A to point B, one must normally specify the path taken.

3. *"Normal" forces produce zero work.* Here "normal forces" refer to any force which is perpendicular to the path. Recall that the vectors $d\vec{\boldsymbol{r}}$ are tangent to the path as depicted in Figure 2.6. If the force is perpendicular to the path, then the dot product is zero. If the force is perpendicular over the entire path between points A and B, then the integral must be zero.

Physical dimensions: From the dot product in (2.8), we see that work is a force times a distance. Since force has dimensions $\mathrm{M\,L/T^2}$, the dimensions for work are $\mathrm{M\,L^2/T^2}$.

Units of measure: Typically, when quantifying mechanical work, one uses units of foot-pounds (ft lbs) or Newton-meters (N m). Also, one 1 Newton-meter is equivalent to a Joule (J).

2.2.1 Work Due to a Constant Force

In general, the work integral in (2.8) can be difficult to calculate. However, there are cases for which it can be very simple. Consider, for example, the rocket car shown in in Figure 2.7. Suppose that the rocket produces a constant thrust force $\vec{\boldsymbol{F}}$, downward and to the right, as the car moves horizontally, to the right.

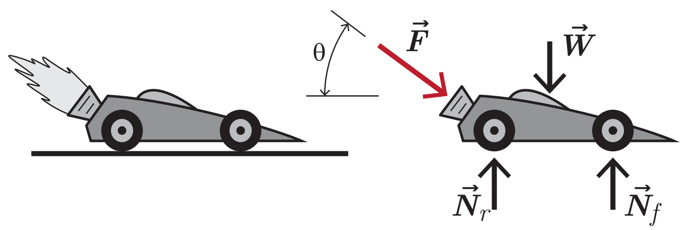

Figure 2.7: Rocket car with constant thrust force $\vec{\boldsymbol{F}}$.

Note that for this problem, the $d\vec{\boldsymbol{r}}$ vectors are pointed directly to the right, since the path of the car is horizontal and directed to the right. Therefore the dot products involving the normal forces, $\vec{\boldsymbol{N}}_r \cdot d\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{N}}_f \cdot d\vec{\boldsymbol{r}}$ are both zero since they act perpendicular to the path. (See observation 3 on the previos page.) Furthermore, the dot product involving the thrust force, $\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$, is *constant* since $\vec{\boldsymbol{F}}$ is constant in both magnitude and direction.

If we write $d\vec{\boldsymbol{r}} = ds\,\hat{\boldsymbol{\imath}}$, then the work integral (2.8) for thrust the thrust force becomes

$$\begin{aligned} U_{A\to B} = \int_A^B \vec{\boldsymbol{F}} \cdot ds\,\hat{\boldsymbol{\imath}} &= \int_{s_A}^{s_B} \left(\vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{\imath}}\right) ds \\ &= \int_{s_A}^{s_B} |\vec{\boldsymbol{F}}|\,\cos(\theta)\,ds = |\vec{\boldsymbol{F}}|\,\cos(\theta)\,s\Big|_{s_A}^{s_B} \\ &= |\vec{\boldsymbol{F}}|\cos(\theta)\,(s_B - s_A)\,. \end{aligned}$$

Here, $|\vec{\boldsymbol{F}}|\cos(\theta)$ is the component of the thrust force tangent to the path. It is the piece that does work. And if the two endpoints A and B are separated by a distance d, then $s_B - s_A = d$, and

$$\boxed{U_{A\to B} = |\vec{\boldsymbol{F}}|\,\cos(\theta)\,d.} \tag{2.10}$$

It is important to note here that the reason this integral is so easy to calculate is *not* only because $\vec{\boldsymbol{F}}$ is constant, but rather that $\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ is constant.

Whenever this happens, (2.10) tells us that work is simply the component of the force tangent to the path multiplied by the distance traveled in going from A to B.

2.2.2 Work Performed by Gravity

The force of gravity, near the surface of the Earth, is usually modeled as a constant force: $\vec{\boldsymbol{W}} = -mg\,\hat{\boldsymbol{\jmath}}$. Since weight is constant, one may be inclined to treat as a constant force as discussed in the previous Section, 2.2.1. Indeed, if the path the object takes a straight line, you can calculate the work as a product of the tangential component of the force and the distance traveled. However, if the path is *not* a straight line, the dot product $\vec{\boldsymbol{w}} \cdot d\vec{\boldsymbol{r}}$ is *not* constant and thus one *cannot* employ the result of the previous section.

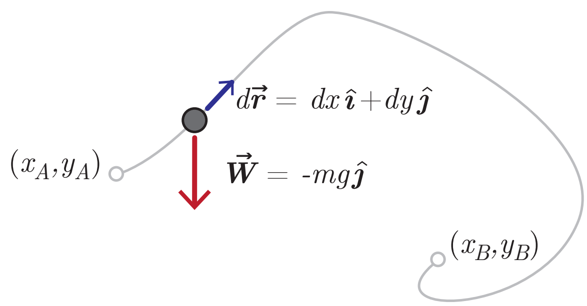

Figure 2.8: A path over which to calculate work by gravity.

Instead, let's take a fresh look at the work calculation. Figure 2.8 shows a hypothetical path. Here, we write the $d\vec{\boldsymbol{r}}$ vector as $d\vec{\boldsymbol{r}} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}}$. Given this, we can express the work due to gravitational force as

$$
\begin{aligned}
U_{A\to B} = \int_A^B \vec{\boldsymbol{W}} \cdot d\vec{\boldsymbol{r}} &= \int_A^B \left(-mg\,\hat{\boldsymbol{\jmath}} \cdot (dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}})\right) \\
&= \int_A^B \left((-mg\,\hat{\boldsymbol{\jmath}} \cdot dx\,\hat{\boldsymbol{\imath}}) + (-mg\,\hat{\boldsymbol{\jmath}} \cdot dy\,\hat{\boldsymbol{\jmath}})\right) \\
&= \int_{y_A}^{y_B} -mg\,dy.
\end{aligned}
$$

On the second line above, we get two dot product. The first is zero because the two vectors are perpendicular to each other. In the second dot product, the two vectors are both vertical. As a result, the line integral becomes a simple integral of a constant over the vertical coordinate, y. Continuing the calculation, we get

$$U_{A\to B} = \int_{y_A}^{y_B} -mg\,dy = -mg\,y\Big|_{y_A}^{y_B} = -mg\,(y_B - y_A)\,.$$

So the work performed by gravity is

$$\boxed{U_{A\to B} = mg\,(y_A - y_B)\,.} \tag{2.11}$$

Observations: There are some profound implications of this result:

- First, observe that the total work done by gravity over the path shown in Figure 2.8 only depends upon the endpoints. Therefore, it does not matter which path the object takes in going from point A to point B. All paths between the two points lead to the same amount of work. This is not true for most other forces. But, *for gravity, work is path independent.*

- The work performed by gravity is simply the product of the weight (mg) and the change in height of the two endpoints ($y_A - y_B$).

2.2.3 Work due to a Simple Spring Force

Next, let's calculate the work of a spring pushing an object along a straight line. In Figure 2.9 below, we consider a bead that can slide left and right along a frictionless wire. We denote the location of the bead along the wire with the variable s. At $s = 0$, the spring is at its natural length. For positive values of s, as shown in the figure, the spring is stretch and pulls to the left (negative $\hat{\boldsymbol{\imath}}$ direction). When s is negative (not shown), the spring is compressed, and pushes to the right.

Referring back to Section 2.1.2, we can write the force of the spring acting on the bead as

$$\vec{\boldsymbol{F}}_s = -ks\,\hat{\boldsymbol{\imath}}.$$

Now if we write $d\vec{\boldsymbol{r}}$ as $ds\,\hat{\boldsymbol{\imath}}$, the work produced by the spring as the object moves from $s = 0$ to $s = s_A$ is

$$\begin{aligned} U_{0\to A} &= \int_0^A \vec{\boldsymbol{F}}_s \cdot d\vec{\boldsymbol{r}} = \int_0^A -ks\,\hat{\boldsymbol{\imath}} \cdot ds\,\hat{\boldsymbol{\imath}} = \int_0^{s_A} -ks^2\,ds. \\ &= -\frac{1}{2}ks^2\Big|_0^{s_A} = -\frac{1}{2}ks_A^2. \end{aligned} \tag{2.12}$$

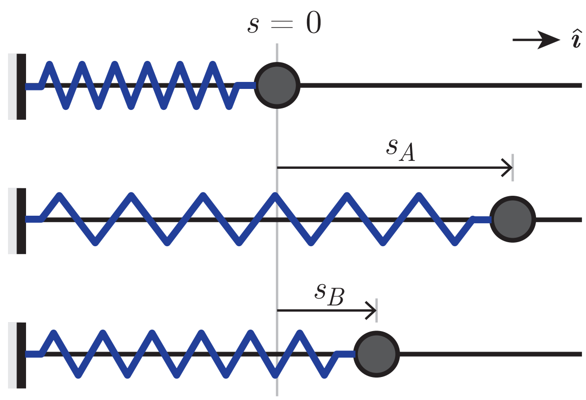

Figure 2.9: Spring acting on an object as it moves along a straight line.

Observations: Here are some observations about the work performed by a spring:

- First, in this case, notice that the work produced by the spring is negative. Why? Notice that as the bead moves to the right from 0 to s_A, the spring is pulling to the left. Since displacement and force are in opposite directions, the work must be negative.

- Also observe that if the bead had moved in the opposite direction from 0 to $-s_A$, the amount of work would be exactly the same:

$$U_{0 \to -s_A} = -\frac{1}{2}k(-s_A)^2 = -\frac{1}{2}ks_A^2 = U_{0 \to +s_A}.$$

- From (2.12), we can see what the spring's work would be if the bead moved from location $s = s_A$ to location $s = s_B$:

$$\boxed{U_{A \to B} = \frac{1}{2}k\left(s_A^2 - s_B^2\right) = \frac{1}{2}ks_A^2 - \frac{1}{2}ks_B^2.} \tag{2.13}$$

  Notice in Figure 2.9 that s_A is larger than s_B. Therefore the work given by (2.13) is positive. This is what we expect since the spring is pulling to the left in this case, and the bead is moving in the same direction.

- Next, let's consider the work by the spring moves from $s = 0$ to $s = s_B$ via an indirect path in which it first moves from $s = 0$ to $s = s_A$ and then moves from $s = s_A$ to $s = s_B$. In this case, the work would be

$$U_{0\to B} = U_{0\to A} + U_{A\to B} = \left(-\frac{1}{2}ks_A^2\right) + \left(\frac{1}{2}ks_A^2 - \frac{1}{2}ks_B^2\right) = -\frac{1}{2}ks_B^2.$$

This is exactly the same amount of work we would get if the path went from $s = 0$ to $s = s_B$ directly.

This is profound. Just like gravity, this is telling us that the work done by a spring is path independent. Most forces do not have this property. We will discuss this property more in the next section.

2.2.4 Conservative Forces

To do.

2.3 Power

Power is the rate at which work is performed. We can write it as

$$P = \frac{dU}{dt}. \tag{2.14}$$

Since work is a scalar, power must be a scalar too. But since work (2.8) is an integral over a path from some beginning point A to some end point B, it might be a little difficult to wrap your head around what (2.14) means. After all, work is an integral over *space* and power is a derivative of work with respect to *time*.

The quickest way to resolve this is to turn work's path integral into a time integral:

$$U_{A\to B} = \int_A^B \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = \int_{t_A}^{t_B} \vec{\boldsymbol{F}} \cdot \left(\frac{d\vec{\boldsymbol{r}}}{dt}\, dt\right) = \int_{t_A}^{t_B} \left(\vec{\boldsymbol{F}} \cdot \frac{d\vec{\boldsymbol{r}}}{dt}\right) dt.$$

But since $\frac{d\vec{\boldsymbol{r}}}{dt}$ is the velocity of the object as it moves along the path, we get

$$U_{A\to B} = \int_{t_A}^{t_B} \left(\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}\right) dt. \tag{2.15}$$

Now, using the fundamental theorem of calculus, we find, at any portion along the path

$$\boxed{P = \frac{dU}{dt} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}.} \tag{2.16}$$

This is the more common way of writing power. This also gives us a new way of writing work; you'll see that (2.15) is equivalent to

$$U_{A \to B} = \int_{t_A}^{t_B} P \, dt. \tag{2.17}$$

That is, we can think of work as the time integral of power.

Physical dimensions: Power, by definition, is work per unit time. As described in Section 2.2, work has dimensions $\mathrm{M\,L^2/T^2}$. Therefore, Power has dimensions $\mathrm{M\,L^2/T^3}$.

Units of measure: In SI units, power can be expressed as a Joule per second, also called a Watt. In wacky imperial units, Power is often expressed in units of "horsepower" which is equivalent to 550 foot-pounds per second or 745.7 Watts. Incidentally, an actual horse can produce about 15 horsepower.

2.4 Impulse

More to come.

2.5 Moment

More to come.

Chapter 3

Kinetics (of Particles)

Kinetics refers to the effect of forces and moments on the motion of bodies that have mass. In this chapter, we will consider bodies that can be represented by a single concentrated mass.

3.1 Newton's Second Law

This is the primary physical principle of Newtonian mechanics. All other physical principles we will discuss in this class can be derived from this one.

$$\boxed{\sum \vec{\boldsymbol{F}} \rightarrow m\vec{\boldsymbol{a}}.} \tag{3.1}$$

Normally, this *equation* is written with an equal sign (=) instead of a rightward pointing arrow (→). However, I want you to think of the equation as a story. By placing an arrow in the equation above, I'm stating that I want you think of this as a *cause and effect relationship.* Specifically, I want you think of net force as something that *causes* an acceleration.

Physical dimensions: Both sides Newton's Second Law relationships above have dimensions $\frac{ML}{T^2}$.

3.1.1 Newton's Second Law, Momentum Edition

In the *Principia*, Newton's historic book in which he presented his theories on how the physical world works, he didn't actually state $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. Instead, he wrote:

> The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

To unpack what this means, we have to understand the word "motion." Here, Newton writes:

> The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjunctly.

The "quantity of matter" is what we call *mass*. So the "quantity of motion" is the product of mass and velocity. Today, we call this linear momentum,

$$\vec{\boldsymbol{L}} = m\vec{\boldsymbol{v}}. \tag{3.2}$$

The "alteration of motion" Newton talks about is the rate that momentum is changing in time. Newton, the inventor of calculus, is talking about the time derivative[1] of momentum. Therefore, to write Newton's Second Law mathematically in the way he understood it, we could write:

$$\boxed{\sum \vec{\boldsymbol{F}} \rightarrow \frac{d\vec{\boldsymbol{L}}}{dt}.} \tag{3.3}$$

Note that

$$\frac{d\vec{\boldsymbol{L}}}{dt} = \frac{d}{dt}\left(m\vec{\boldsymbol{v}}\right) = m\frac{d\vec{\boldsymbol{v}}}{dt} = m\vec{\boldsymbol{a}},$$

since mass is constant. Therefore, the physical principle expressed in (3.3) is mathematically equivalent to (3.1).

Physical Dimensions: Linear momentum $\vec{\boldsymbol{L}}$, is a mass times a velocity. Therefore, it has dimensions of $\frac{M\,L}{T}$.

3.2 Angular Momentum Principle

Suppose we have an object of mass m moving with velocity $\vec{\boldsymbol{v}}$ as indicated in Figure 3.1. Also, let $\vec{\boldsymbol{r}}$ denote the position vector of the object relative to some origin we call o. Given these, we can define the angular momentum

[1] Newton didn't use the word "derivative." That came later. Instead he used the word "fluxion."

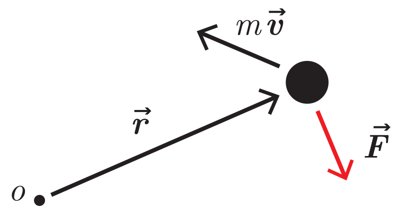

Figure 3.1: Elements of the angular momentum.

about point o as

$$\vec{H}_o = \vec{r} \times \vec{L} = \vec{r} \times (m\,\vec{v})\,. \tag{3.4}$$

Here, $\times$ indicates a vector cross product. Therefore $\vec{H}_o$ is a vector. If the position $\vec{r}$ and velocity $\vec{v}$ both lie in the plane of this page, then the angular momentum vector $\vec{H}_o$ is perpendicular to the page. It is pointed out of the page in this case.

Also shown in Figure 3.1 is a force $\vec{F}$ acting on the object. We can think of it as the net force acting on the object. Then $\vec{r} \times \vec{F}$ is the moment $\vec{M}_o$ of the force about point o. If force $\vec{F}$ lies in the plane of the page, then the moment is perpendicular to the page, pointed inward.

Given these definitions, the Angular Momentum Principle states:

$$\boxed{\vec{M}_o \to \frac{d\,\vec{H}_o}{dt}.} \tag{3.5}$$

Again, I use the right arrow ($\to$) to denote causality. I like to think of the moment *causing* a change in angular momentum.

Derivation.

So where does (3.5) come from? Let's start with the definition of angular momentum (3.4) and take a derivative.

$$\frac{d\,\vec{H}_o}{dt} = \frac{d\,\vec{r}}{dt} \times (m\vec{v}) + \vec{r} \times \left(m\frac{d\,\vec{v}}{dt}\right).$$

Here, we are using the product rule for derivatives. Recognizing that $\frac{d\vec{r}}{dt}$ is velocity and $\frac{d\vec{v}}{dt}$ is acceleration, we can write this expression as

$$\frac{d\,\vec{H}_o}{dt} = \vec{v} \times (m\vec{v}) + \vec{r} \times (m\vec{a})\,.$$

Note that the vectors $\vec{\boldsymbol{v}}$ and $m\vec{\boldsymbol{v}}$ are in the same direction. Therefore, the first cross product above is zero. Because of Newton's Second Law, we can replace the term $m\vec{\boldsymbol{a}}$ with $\vec{\boldsymbol{F}}$. This gives

$$\frac{d\,\vec{\boldsymbol{H}}_o}{dt} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}},$$

which is equivalent to (3.5).

3.3 Work-Energy Principle

The Work-Energy Principle states:

$$\boxed{U_{A\to B} \to T_B - T_A.} \tag{3.6}$$

Here, $U_{A\to B}$ is the work performed by all forces acting on an object as it moves along a specified path from some beginning point A to some end point B. As described in Section 2.2, work is defined by a path integral: $U_{A\to B} = \int_A^B \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$. In that section, we showed how to calculate the work integral for gravitational forces, spring forces and constant forces. The terms $T_A = \frac{1}{2}mv_A^2$ and $T_B = \frac{1}{2}mv_B^2$ are the kinetic energies at the beginning and end points of the path, respectively.

Again, I put the right arrow ($\to$) in (3.6) just to emphasize a point of view that work *causes* a change in kinetic energy. When I use the Work-Energy Principle to solve problems, I usually use an equal sign (=) because it leads to equations I must solve, in which the stuff on the left side is equal to the stuff on the right.

Derivation.

More to come.

3.4 Impulse Momentum Principle

More to come.

Part II

Systems of Particles and Rigid Bodies

Chapter 4

Kinematics of Systems of Particles and Rigid Bodies

So far, we have been discussing the motion of single, individual particles. Now we are going to start discussing the motion of a collection of particles. The particles might be free and move independently of each other. Or, the particles might be stuck together to form a body.

4.1 System of Particles, Center of Mass

Suppose we have a collection of particles. We will call it a "system" of particles. To keep things simple, we will start by considering just three particles. Everything we do here can be extended to any number of particles. The three particles are shown in Figure 4.1, along with their position vectors: $\vec{\boldsymbol{r}}_1$, $\vec{\boldsymbol{r}}_2$, and $\vec{\boldsymbol{r}}_3$.

4.1.1 Averaging the Position Vectors

If you three scalar quantities, for example, the three masses of the particles (m_1, m_2, m_3), you would know how to calculate the *average* or *mean* of these quantities. One could compute the average mass, m_{avg}, as

$$m_{avg} = \frac{1}{3}\left(m_1 + m_2 + m_3\right).$$

The average is defined by adding the quantities together, and then multiplying by a scalar, $\frac{1}{3}$.

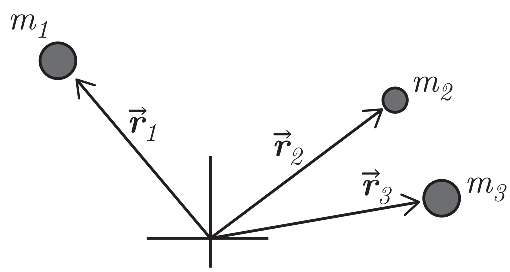

Figure 4.1: A system of three particles.

Now, since vector addition is well defined, as is multiplication of a scalar with a vector, it is possible to define the "average" position in a similar way:

$$\vec{\boldsymbol{r}}_{avg} = \frac{1}{3}\left(\vec{\boldsymbol{r}}_1 + \vec{\boldsymbol{r}}_2 + \vec{\boldsymbol{r}}_3\right). \tag{4.1}$$

To see what this means, let's decompose the position vectors into components, *e.g* $\vec{\boldsymbol{r}}_1 = x_1\,\hat{\boldsymbol{\imath}} + y_1\,\hat{\boldsymbol{\jmath}} + z_1\,\hat{\boldsymbol{k}}$. Substituting the components into (4.1), we get

$$\vec{\boldsymbol{r}}_{avg} = \frac{1}{3}\left(\left(x_1\,\hat{\boldsymbol{\imath}} + y_1\,\hat{\boldsymbol{\jmath}} + z_1\,\hat{\boldsymbol{k}}\right) + \left(x_2\,\hat{\boldsymbol{\imath}} + y_2\,\hat{\boldsymbol{\jmath}} + z_2\,\hat{\boldsymbol{k}}\right) + \left(x_3\,\hat{\boldsymbol{\imath}} + y_3\,\hat{\boldsymbol{\jmath}} + z_3\,\hat{\boldsymbol{k}}\right)\right).$$

Or,

$$\vec{\boldsymbol{r}}_{avg} = \frac{1}{3}\left(x_1 + x_2 + x_3\right)\hat{\boldsymbol{\imath}} + \frac{1}{3}\left(y_1 + y_2 + y_3\right)\hat{\boldsymbol{\jmath}} + \frac{1}{3}\left(z_1 + z_2 + z_3\right)\hat{\boldsymbol{k}}. \tag{4.2}$$

Interpretation. Therefore, the interpretation of the average position vector, $\vec{\boldsymbol{r}}_{avg}$, is quite straightforward. The $\hat{\boldsymbol{\imath}}$ component of $\vec{\boldsymbol{r}}_{avg}$ is simply the average of x_1, x_2, a x_3, the $\hat{\boldsymbol{\imath}}$ components of the original position vectors. The $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ components of $\vec{\boldsymbol{r}}_{avg}$ work similarly.

This average position vector $\vec{\boldsymbol{r}}_{avg}$ given by (4.1) generally does *not* have physical significance because it treats each mass equally. That is, each particle has the same $\frac{1}{3}$ contribution to the average, regardless of their relative mass. If one of the particles is much more massive than the other two, then it should have a larger contribution to a physically meaningful average.

4.1.2 Center of Mass, A Weighted Average

To create a more physically meaningful "average" position for the system of particles, we need to weight the average so that larger particles have a proportionally larger contribution to the result. The result is called the ***center of mass***, $\vec{\boldsymbol{r}}_G$. For three particles in Figure 4.1 with masses m_1, m_2, and m_3, the center of mass is given by

$$\vec{\boldsymbol{r}}_G = \frac{m_1\,\vec{\boldsymbol{r}}_1 + m_2\,\vec{\boldsymbol{r}}_2 + m_3\,\vec{\boldsymbol{r}}_3}{m_1 + m_2 + m_3}. \tag{4.3}$$

Interpretation. Notice that the denominator is the total mass, m_{tot}. Given that, we can write (4.3) as $\vec{\boldsymbol{r}}_G = \kappa_1\,\vec{\boldsymbol{r}}_1 + \kappa_2\,\vec{\boldsymbol{r}}_2 + \kappa_3\,\vec{\boldsymbol{r}}_3$. Here, $\kappa_j = m_j/m_{tot}$; it's the fraction of the total mass that can be attributed to particle j. Note that if all three particles had the same mass, then the coefficients would be $\kappa_1 = \kappa_2 = \kappa_3 = \frac{1}{3}$, and the center of mass in (4.3) would be equivalent to the average in (4.1). When the masses are different, the particles with greater mass have a larger impact on the the weighted average.

Components.

If we decompose the vectors into components, as we did in Section 4.1.1, $\vec{\boldsymbol{r}}_j = x_j\,\hat{\boldsymbol{\imath}} + y_j\,\hat{\boldsymbol{\jmath}} + z_j\,\hat{\boldsymbol{k}}$, then our expression for center of mass (4.3) can be written as:

$$\begin{aligned}\vec{\boldsymbol{r}}_G =& \frac{m_1\,x_1 + m_2\,x_2 + m_3\,x_3}{m_{tot}}\hat{\boldsymbol{\imath}} + \frac{m_1\,y_1 + m_2\,y_2 + m_3\,y_3}{m_{tot}}\hat{\boldsymbol{\jmath}} + \\ & \frac{m_1\,z_1 + m_2\,z_2 + m_3\,z_3}{m_{tot}}\hat{\boldsymbol{k}}.\end{aligned} \tag{4.4}$$

Therefore, each component of the center of mass vector is a weighted average of the corresponding components of the position vectors. I encourage students to

4.2 Relative Motion

More to come

4.3 A Rigid Body

If an object is *not deforming* and *not rotating*, then all of its mass has the same velocity and the same acceleration. Thus, all of the mass acts as one. In such cases, one can treat the dynamics of the object as a single particle occupying a single point in space at any one time.

However, if an object's rotation is an important part of its dynamics, then we will have to consider how its mass is distributed in space and how those mass particles that make up the body are connected to each other. Here, we discuss the simplest type of object with distributed mass, a rigid body.

4.3.1 Definitions

> **Definition:** A ***rigid body*** is a collection of particles for which the distance between any two particles is constant.

One way to think of a rigid body is a collection of discrete particles connected by massless rods of fixed length as illustrated in Figure 4.2. Since the lengths of the rods do not change, the distance between any two particles does not change while the collection of particles moves and rotates. It is a *rigid body*.

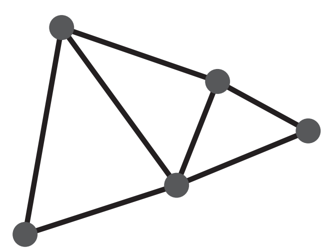

Figure 4.2: Rigid body consisting of discrete particles connected by massless rigid rods.

Yet, most solid objects we encounter on a day-to-day basis are ones for which mass is distributed continuously, right? As one holds a piece of steel in one's hand, for example, it looks looks like a continuous chunk of metal. In these notes, I will use the bean-shaped object shown in Figure 4.3 to denote a generic body in which mass is distributed continuously. Specifically, in the figure, we depict the body at two different times. Between those two times, the body has moved.

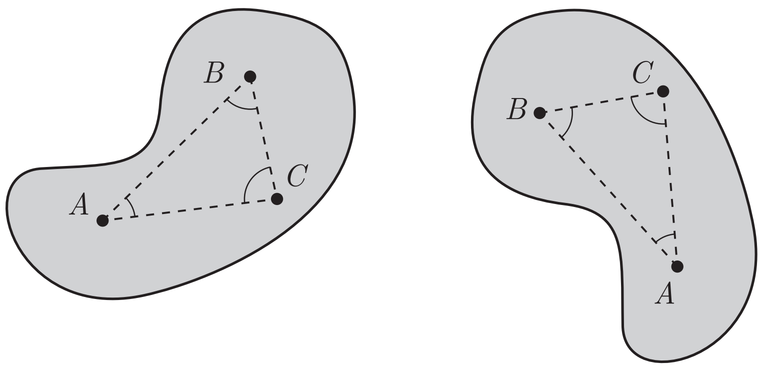

Figure 4.3: Rigid body with a continuous distribution of mass.

When describing how a rigid body moves, we often identify points that are fixed to the body, and express their motion. In Figure 4.3, we have three such points labeled A, B, and C, that form the vertices of a triangle. According to the definition at the top of this section, the distances between any pair of these points must NOT change as the body moves. As a consequence, we can make the following observations.

Observations:

1. As the body moves, any triangle drawn on the body maintains its shape.
2. The lengths of the sides of the triangle remain constant.
3. The interior angles of the triangle remain constant.

More to come....

Chapter 5

Kinetics of Systems of Particles and Rigid Bodies

There's a lot to write here.... which will take time. In the meantime, let me express the main ideas mathematically.

5.1 Systems of Particles

The following are true about the dynamics of a system of particles. These relationships come directly from Newton's Second and Third Laws.

$$\sum \vec{\boldsymbol{F}}^{(ext)} = m_{tot}\,\vec{\boldsymbol{a}}_G. \tag{5.1}$$

Interestingly, only the external forces matter in this formulation. All the internal forces (forces between particles) cancel out. Here, m_{tot} is the total mass, and $\vec{\boldsymbol{a}}_G$ is the acceleration of the center of mass.

Also, there is a relationship for the moment:

$$\sum \vec{\boldsymbol{M}}_o^{(ext)} = \frac{d}{dt}\sum \vec{\boldsymbol{H}}_o. \tag{5.2}$$

Again, only moments due to external forces matter.

Resources.

To see derivations and retrieve notes, click on the links below.

- *Definition of Center of Mass.*
- *Newton's Second Law for a System of Particles.*
- *Moment Equation for a System of Particles.*

5.2 Planar Dynamics of Rigid Bodies

A rigid body is a special case of a system of particles. Therefore, the dynamics of the center of mass of a rigid body are the same as (5.1)

$$\sum \vec{\boldsymbol{F}} = m\,\vec{\boldsymbol{a}}_G. \tag{5.3}$$

Here, m is the mass of the rigid body, and $\vec{\boldsymbol{a}}_G$ is the acceleration of the center of mass.

The moment equation can take different forms depending on the rigid body and the point we choose to take moments about. The simplest case is a rigid body that may rotate about a fixed point I call p. In this case, if we choose to take moments about point p, then

$$\sum \vec{\boldsymbol{M}}_p = I_p\,\vec{\boldsymbol{\alpha}}. \tag{5.4}$$

Here, I_p is the moment of inertia about point p and $\vec{\boldsymbol{\alpha}}$ is the angular acceleration of the body.

Another option is to take moments about the center of mass. Here, we get

$$\sum \vec{\boldsymbol{M}}_G = I_G\,\vec{\boldsymbol{\alpha}}. \tag{5.5}$$

I_G, of course, is the moment of inertia about the center of mass.

Finally, if you choose to take moments about some arbitrary point q, the most general moment equation is

$$\sum \vec{\boldsymbol{M}}_q = \vec{\boldsymbol{r}}_{G/q} \times m\,\vec{\boldsymbol{a}}_q + I_q\,\vec{\boldsymbol{\alpha}}. \tag{5.6}$$

In this expression $\vec{\boldsymbol{r}}_{G/q}$ is the position of the center of mass relative to point q. It is not difficult to show that (5.4) and (5.5) are special cases of (5.6).

Resources.

To see derivations and retrieve notes, click on the links below.

- *How the General Moment Equation is Derived.*
- *Moment of Inertia and Parallel Axis Theorem.*