Appendix B

Review of Some Calculus Fundamentals

In Dynamics, objects are moving. This means that positions and velocities are changing with respect to time. The rates at which these things change depend on forces acting on a body. Therefore, to describe what is happening mathematically, we need *calculus.* In fact, calculus was invented by Newton in order to handle dynamics problems.

Here, it is not enought to know how to calculate an integral or derivative. It is *really* important that you have a fundamental understanding of what these concepts *mean.*

B.1 Derivatives (Video)

Suppose we have a function that I'll call $x(t)$. If we think of the independent variable t as time, then $x(t)$ tells us how x varies with time. An example is shown in Figure B.1.

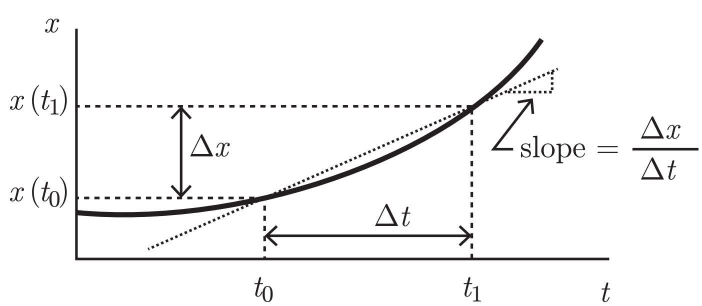

Figure B.1: Plot of a function $x(t)$ to illustrate what a derivative means.

By definition, the derivative of x with respect to t, at time $t = t_0$ is

$$\left.\frac{dx}{dt}\right|_{t_0} = \lim_{t_1 \to t_0} \frac{x(t_1) - x(t_0)}{t_1 - t_0}. \tag{B.1}$$

Now, Equation (B.1) is a collection of symbols. Rather than trying to memorize the symbols and the order in which they appear, it is important to understand what it *means.*

B.1.1 Derivative is a Slope

Notice that both the numerator and denominator are differences (subtraction). The numerator is a measure of how much x has *changed* between times t_0 and t_1. The change is labeled Δx in Figure B.1. The denominator is the *change* in time, labeled Δt. In Figure B.1, we see that the quotient $\Delta x/\Delta t$ is the *slope* of the line passing through the points $x(t_0)$ and $x(t_1)$.

If we choose a different point t_1 (call it t_{1b}), then we get a different line passing through points $x(t_0)$ and $x(t_{1b})$, with a different slope. For comparison, the original slope and new slope are illustrated in Figure B.2. In the limit as t_1 approaches t_0, these slopes approach a limit: the slope

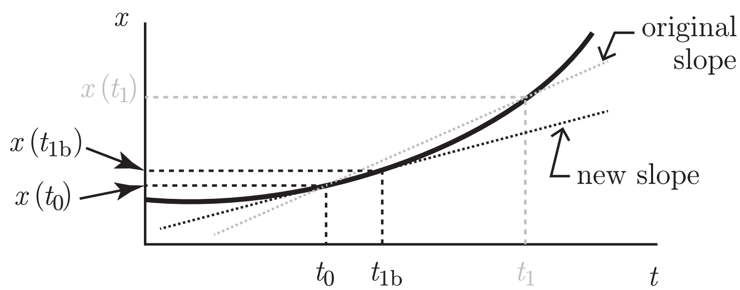

Figure B.2: Illustration of how the slope changes as $t_1 \to t_0$.

of the line tangent to $x(t)$, at the point $t = t_0$. According to the definition (B.1), this limiting slope is the derivative of x with respect to t, at the point $t = t_0$.

B.1.2 Derivative is a Rate of Change

As discussed in the previous section, the quotient $\Delta x/\Delta t$ can be interpreted, graphically, as a slope. We can also think of it as the amount x **changes** (Δx) for a given change in time: Δt.

As an example to think about, suppose that $x(t)$ is the reading on your car's odometer as you're driving across the country. Suppose that t_0 coincides with 9:00am on Wednesday morning, and t_1 coincides with the same time the next day. Furthermore, suppose that during this 24-hour period, your car travels 600 miles. Then,

$$\frac{\Delta x}{\Delta t} = \frac{x(t_1) - x(t_0)}{t_1 - t_0} = \frac{600\,\text{mi}}{\text{day}} = \frac{600\,\text{mi}}{\text{day}} \cdot \frac{\text{day}}{24\,\text{hour}} = 25\,\text{mph}.$$

Thus we can see that $\Delta x/\Delta t$ is a rate of travel; your odometer is changing at a rate of 600 miles per day. Sounds like a good pace doesn't it?

However, when we convert 600 miles per day to miles per hour, we get 25 mph. Hmmm. It doesn't seem so fast now. The reason it doesn't seem so fast is because your 600 miles per day calculation includes the time you were stopped to get fuel, stopped to get food, stopped to sleep, and stopped to look over the rim of the Grand Canyon.

Now, suppose we define t_{1b} to coincide with 10:00am on Wednesday (one hour after t_0), a time during which you were on the interstate highway. Then, $\Delta x_b/\Delta t_b$ would probably provide a better estimate of your rate of travel (speed) during this time. However, this new estimate would still include the time you got stuck behind a slow moving truck and were not able to pass.

If you define t_{1c} so that $\Delta t_c = t_{1c} - t_0$ is 1 second, then $\Delta x_c/\Delta t_c$ is probably a real good estimate of your car's speed at 9:00am. Of course, you would need a really precise odometer to make this measurement.

This process of taking smaller and smaller time intervals is precisely the effect of the limit $\lim_{t_1 \to t_0}$ in the definition of derivative (B.1). The derivative is an instantaneous rate of change.

B.1.3 Shorthand Notation for Time Derivative

In dynamics we will be writing a lot of derivatives. Instead of writing them the long way as in the left side of Equation (B.1), we will often use a simpler notation:

$$\dot{x}(t_0).$$

The over-dot simply means derivative with respect to time. A second derivative with respect to time is indicated by two dots: $\ddot{x}(t_0)$. Third derivative: $\dddot{x}(t_0)$.

Now the simple dot notation only represents differentiation with respect to *time*. The dot is NOT used to represent differentiation with respect to any other variable.

B.1.4 Check Your Understanding (Video)

In the top part of Figure B.3, I show a plot of a function $x(t)$. If you *understand* what the time derivative means, you should be able to sketch (by hand) the time derivative $\dot{x}(t)$. Try it before turning the page (where the answer is given). In both plots, make the time axis have the same scale so that you can line up features of the plots. I have drawn light gray vertical lines to help you align things.

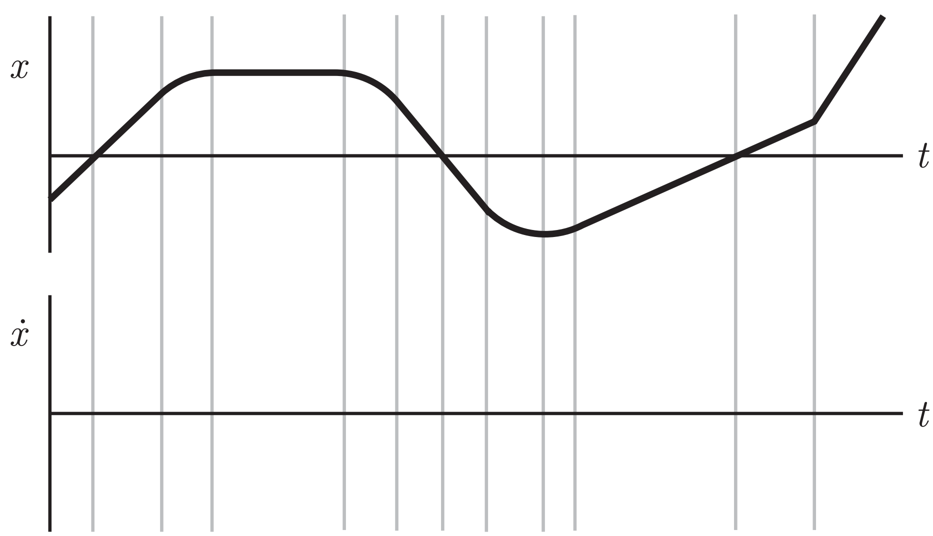

Figure B.3: Test your understanding of the time derivative.

Here's the answer. Examine it very carefully, and read the itemized comments below.

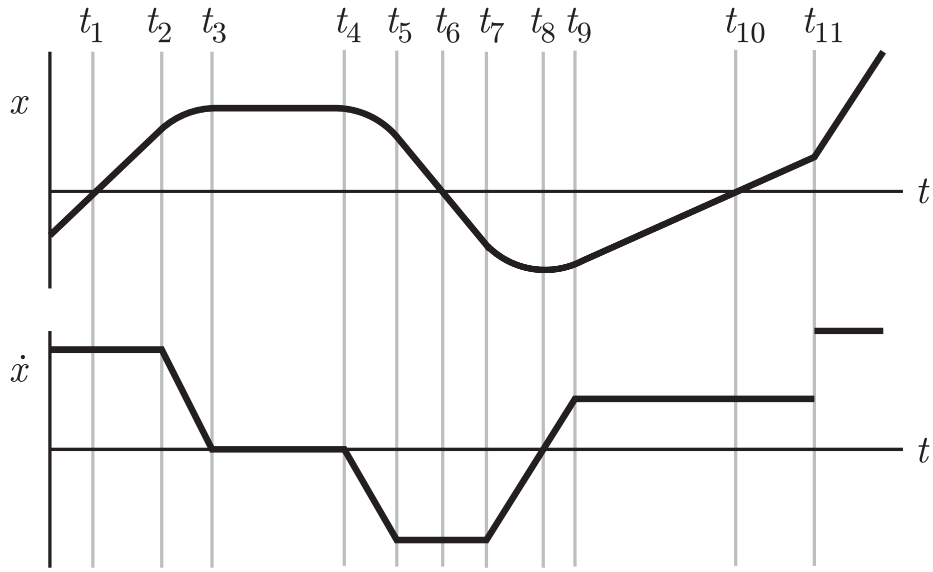

Figure B.4: The answer.

- **Interval** $t_3 < t < t_4$**:** Within this interval, notice that $x(t)$ is constant; its value is not changing. The quantity Δx from Figure B.1 must be zero. Therefore the derivative $\dot{x}(t)$ must be zero between t_3 and t_4.

  If you want to think about it graphically, notice that $x(t)$ has zero slope between t_3 and t_4. Since the value of $\dot{x}(t)$ is the slope of $x(t)$, we know that $\dot{x}(t)$ must be zero between t_3 and t_4.

- **Interval** $0 < t < t_2$**:** Within this interval, notice that $x(t)$ is strictly[1] increasing. The slope of $x(t)$ is positive. Therefore $\dot{x}$ must be positive.

  Furthermore, observe that the slope of $x(t)$ is constant between 0 and t_2. Therefore, in addition to being positive, $\dot{x}(t)$ must be constant in the interval.

  Note: It is common for students to get distracted by the fact that $x(t)$ is negative between 0 and t_1, and then positive between t_1 and t_2. This fact doesn't matter. What does matter is the fact that $x(t)$ is increasing, and that it's *increasing at a constant rate.*

- **Interval** $t_9 < t < t_{11}$**:** Likewise, x(t) is increasing at a constant rate (constant slope) between t_9 and t_{11}. Therefore, $\dot{x}(t)$ must be a positive constant in this interval.

  But notice that the slope between t_9 and t_{11} is not as large as it is between 0 and t_2. Therefore $\dot{x}(t)$ is not as large between t_9 and t_{11} as it is between 0 and t_2.

  Again, the fact that $x(t)$ passes through zero doesn't matter.

- **Interval** $t > t_{11}$**:** Again, $x(t)$ is increasing at a constant rate. The rate (slope) is greater than it is between t_9 and t_{11} and greater than it is between 0 and t_2. Therefore $\dot{x}(t)$ is constant, positive, and larger than it is anywhere else on the graph.

[1]When I say "strictly" increasing, I mean that it only increases; it doesn't decrease or remain constant.

- **Interval $t_5 < t < t_7$:** Here, the function $x(t)$ is decreasing at a constant rate (constant negative slope). Therefore, $\dot{x}(t)$ must be constant and negative between t_5 and t_7.

  To my eye, the *magnitude* of the slope between t_5 and t_7 looks to be about the same as the magnitude between 0 and t_2. Therefore, I have draw $\dot{x}(t)$ so that the magnitudes in the two intervals are about the same.

- **time $t = t_{11}$:** At this instant, notice that the slope of $x(t)$ changes abruptly. It looks like there is a kink in the graph. This is why the derivative $\dot{x}(t)$ changes abruptly at t_{11}.

- **Interval $t_2 < t < t_3$:** This is a region of transition in which the slope of $x(t)$ goes from positive to zero. But it does so continuously. There is no abrupt change in slope, or "kink" in $x(t)$, like there was at t_{11}. Since the slope of $x(t)$ changes continuously from t_2 through t_3, the derivative $\dot{x}(t)$ must vary continuously in this interval.

  Not enough information is given to us in the problem to know exactly how $\dot{x}(t)$ varies between t_2 and t_3. For simplicity, I have used straight lines.

- **Interval: $t_7 < t < t_9$:** This is another interval of transition in which the slope of $x(t)$ continuously goes from a negative value to a positive value. I have drawn my derivative as a straight line in the interval.

  The new, potentially tricky, part about this part is that in going from negative to positive continuously, the derivative $\dot{x}(t)$ must pass through zero. What happens when the derivative is zero?

  When the derivative $\dot{x}(t)$ is zero, the slope of $x(t)$ is zero. Notice that the slope of $x(t)$ is zero for one brief instant, at $t = t_6$. I have draw my plot of $\dot{x}(t)$ so that it passes through zero at t_6.

  Note: The derivative being zero for one brief instant at t_6 is very different from the derivative being zero over a whole time interval $t_3 < t < t_4$. Between t_3 and t_4, the derivative is zero because $x(t)$ is constant; it's not changing at all during the interval. At t_6, the zero derivative does not occur because $x(t)$ stops changing. Rather, it passes through zero as the rate of change goes from negative to positive. We see that as the slope goes from negative to positive, we encounter a local minimum in $x(t)$. If the slope went from positive to negative, we would have a local maximum.

B.1.5 Take-Aways

Here is a list of what I feel are the most important points that you should take away from Section B.1.

1. A time derivative is a measure of how fast something is changing. It is a *time rate of change.*
2. If we plot a function, then the derivative of the function at a time t_0 is the *slope* of that function at $t = t_0$.
3. When something is *increasing* with respect to time, its time derivative is *positive.*
4. When something is *decreasing* with respect to time, its time derivative is *negative.*
5. When something is *not changing*, its time derivative is *zero.*
6. A time derivative of zero does NOT necessarily mean that the function is not changing. A function will momentarily have a *time derivative of zero when it reaches a local peak (maximum) or local minimum*, for example.

B.2 Integration

The other fundamental operation you learned in your first calculus course was integration. I summarize what I believe to be the most important parts (from the perspective of Engineering Dynamics) here.

B.2.1 Integration is the Opposite of Differentiation

Suppose we had an "operator" called +2, which takes any number as an input and produces another number (the output) by adding two to the input. So, for example, $6 \xrightarrow{+2} 8$ or $11 \xrightarrow{+2} 13$. Here, 6 and 11 were the inputs; 8 and 13 were the outputs.

For the operator +2, there is another operator -2 (e.g. $8 \xrightarrow{-2} 6$) which acts as an inverse. In other words, if we have two numbers a and b such that

$$a \xrightarrow{+2} b, \qquad \text{then} \qquad \text{b} \xrightarrow{-2} \text{a.}$$

Similarly, we can define an operator $\times 5$ which multiplies the input by 5: e.g. $6 \xrightarrow{\times 5} 30$. It's probably not a surprise that multiplication also has an *inverse.* It's called division by five: $\div 5$. Therefore,

$$a \xrightarrow{\times 5} b, \qquad \text{then} \qquad \text{b} \xrightarrow{\div 5} \text{a.}$$

Differentiation and integration work in a very similar way. We can think of the derivative as an operator that takes a function $x(t)$ as input and produces another function $\dot{x}(t)$ as output:

$$x(t) \xrightarrow{\text{derivative}} \dot{x}(t).$$

The *inverse* of the derivative operator is the integration operator:

$$\dot{x}(t) \xrightarrow{\text{integration}} x(t).$$

In your first calculus class, you may have called the integral the anti-derivative. It's because integration does the opposite thing as differentiation.

B.2.2 Check Your Understanding (Video)

Back in that first calculus course you probably spent a lot of time learning how to integrate specific functions (e.g. t^2, $\log(t)$, $\cos(2t)$, et cetera). All that is important. However, I want to make sure you have more of an intuitive *understanding* of what integration is.

The exercise I want you to complete below is exactly the opposite of that in Section B.1.4. Now, I'm giving you the function $\dot{x}(t)$ and I want you to integrate it to sketch, by hand, a plot of $x(t)$. You should keep the scale on the time axis the same. I have included some light gray vertical lines in order to help you align features of your plot. Do not turn the page until you have completed your plot.

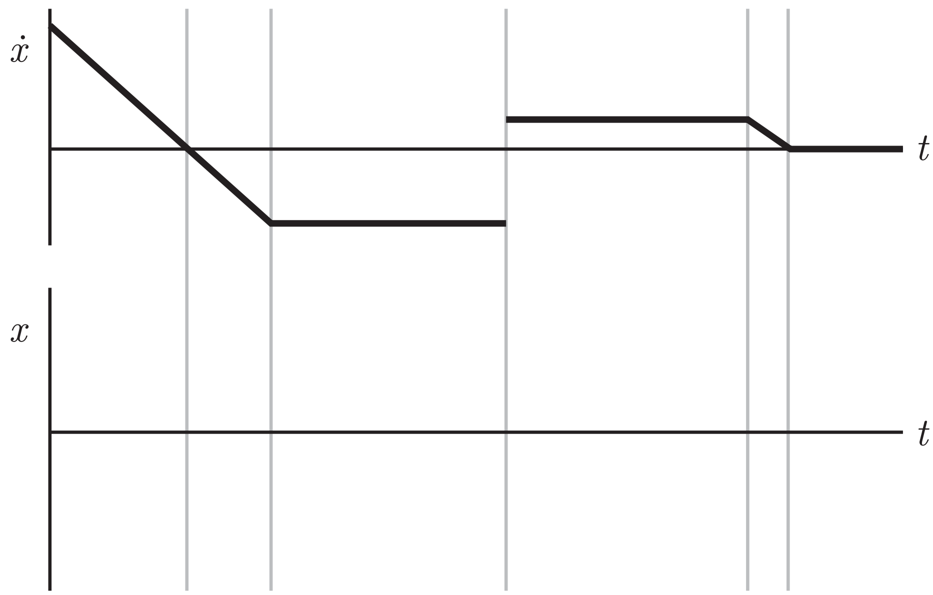

Figure B.5: Test your understanding of the time integration.

Here's an answer[2]. Examine it very carefully, and read the itemized comments below.

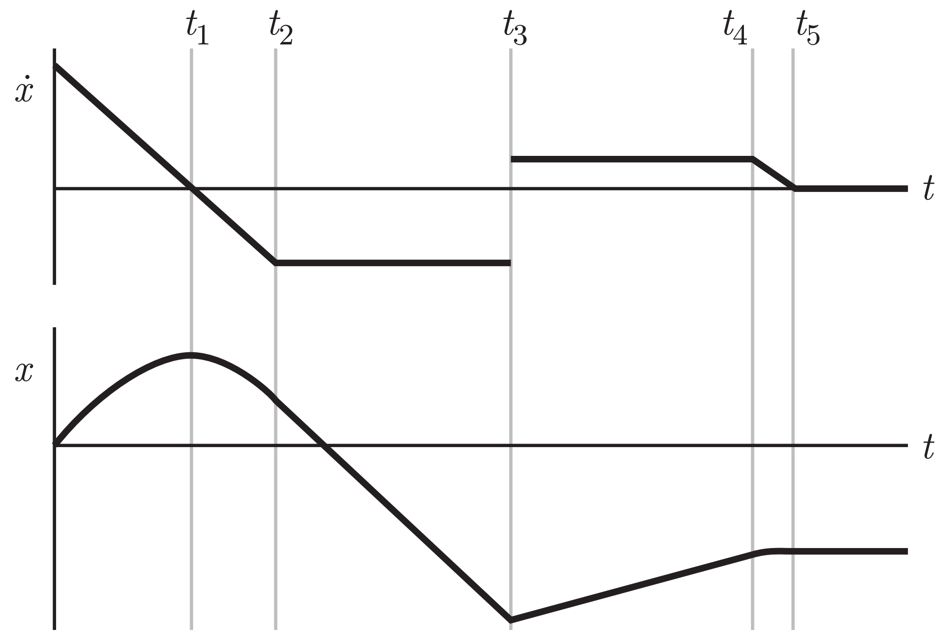

Figure B.6: An answer.

- **Starting point:** The problem statement on the previous page does not tell us where $x(t)$ should begin. For simplicity, I choose $x(0) = 0$. As we shall discuss in Section B.2.3, the choice is arbitrary.

- **Interval** $0 < t < t_1$**:** Because $\dot{x}(t)$ is positive in this interval, $x(t)$ must be increasing. However the rate at which $x(t)$ increases becomes smaller as $\dot{x}(t)$ becomes smaller. Graphically, we see that the slope of $x(t)$ starts off with a relatively steep slope, but the slope gets shallower in time until $t = t_1$, where the slope is zero.

- **Interval** $t_1 < t < t_2$**:** In this interval, we continue the parabolic curve that began in the previous interval. Between t_1 and t_2, the $x(t)$ begins to decrease, and the rate of decrease becomes more pronounced up until $t = t_2$.

- **Instant** $t = t_1$**:** At this instant, the derivative is zero and $x(t)$ reaches a local maximum.

- **Interval** $t_2 < t < t_3$**:** In this interval $\dot{x}(t)$ is negative and constant. Therefore, $x(t)$ decreases at a constant rate; it is a line with constant negative slope. Notice that the slope of this straight line *matches* the slope of the parabola at $t = t_2$, as it must since $\dot{x}(t)$ is continuous at t_2. There is no abrupt jump in the value of $\dot{x}(t)$, and hence no sudden change in slope of $x(t)$.

- **Interval** $t_3 < t < t_4$**:** At $t = t_3$, there is a sudden jump in $\dot{x}(t)$. Therefore we do see a sudden change in the slope of $x(t)$. Now, $\dot{x}(t)$ is positive and constant. Therefore $x(t)$ is a straight line with positive and relatively small slope. $x(t)$ increases at a constant (somewhat small) rate.

[2] Notice that I said "an" answer rather than "the" answer. I'll explain the squishy language in Section B.2.3.

- **Interval** $t_4 < t < t_5$**:** In this interval, $\dot{x}(t)$ drops rather rapidly from something positive to zero. Therefore, the slope of $x(t)$ rapidly – but continuously – rolls off to zero (flat).
- **Interval** $t > t_5$**:** Here, $\dot{x}(t)$ is zero over an interval, meaning that $x(t)$ does not change over that interval. Therefore we see a constant value of $x(t)$ over the interval.

In the discussion above, there are a few aspects I have left off. For example, why is $x(t_3)$ positive? Why is $x(t_5)$ negative? We will address these issues in Section B.2.4.

B.2.3 Constant of Integration

Suppose we take the answer to the previous question and shift it upward a bit, or shift it downward, like one of the dark gray $x(t)$ curves in Figure B.7. At any given time t, each of the $x(t)$ curves has the same slope. Therefore, each curve $x(t)$ is an anti-derivative of the function $\dot{x}(t)$ in the top half of the figure. Thus, there are multiple valid answers to the question; each one is an integral to $\dot{x}(t)$.

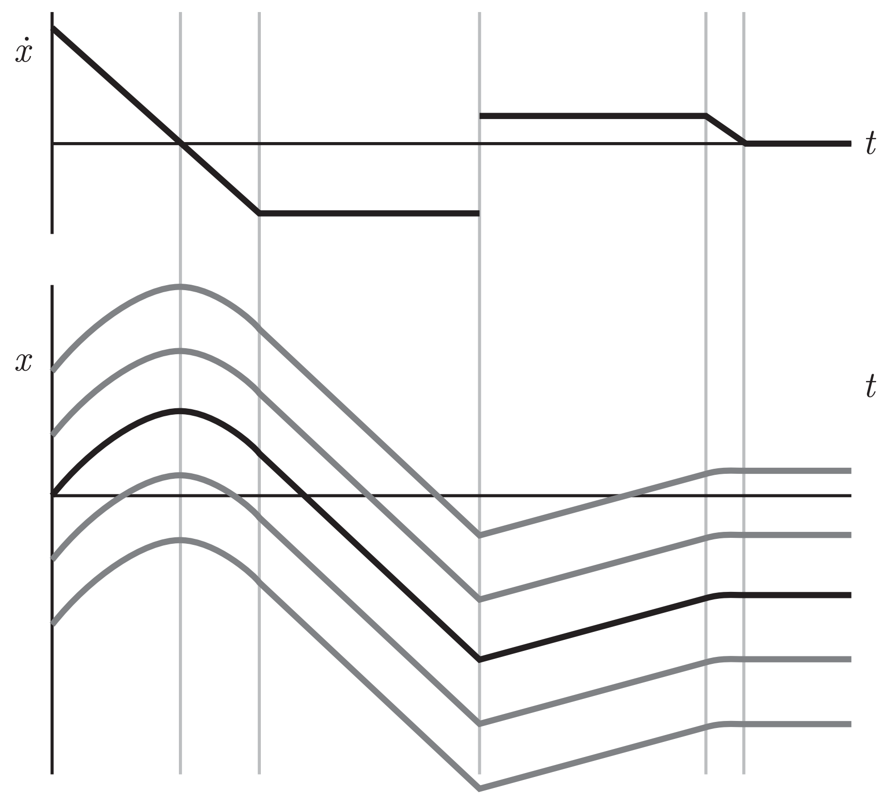

Figure B.7: There are multiple valid answers.

I bet you already knew this. In your first calculus class, you learned how to integrate a function like $f(t) = t^2$. When you did, you got

$$\int t^2\,dt = \frac{1}{3}\,t^3 + c.$$

Similarly,

$$\int \sin(5t)\,dt = -\frac{1}{5}\cos(5t) + c, \qquad \int \frac{1}{t}\,dt = \ln(t) + c, \qquad \int \ln(t)\,dt = t\,\ln(t) - t + c.$$

That last one can can be obtained via integration by parts[3]. The point I'm trying to get you to remember is that each one of these (indefinite) integrals has a "$+c$", an arbitrary constant of integration that can be added to it.

The fact that we get a whole family of answers to the "Check Your Understanding" question in the previous section is because of the constant of integration. The additive constant shifts the curve up or down as depicted in Figure B.7.

B.2.4 Integration is an Area Calculation

You may recall from your first calculus class that integration can also be interpreted as an area under a curve. When we think of integration as area, let's consider it as a definite integral:

$$x(t) = \int_{\tau=0}^{\tau=t} \dot{x}(\tau)\, d\tau. \tag{B.2}$$

Furthermore, we'll consider a specific concrete example shown in Figure B.8. In the figure, a function $\dot{x}(\tau)$ that we wish to integrate is given. The light gray vertical lines mark off six equally spaced times t_1 through t_6.

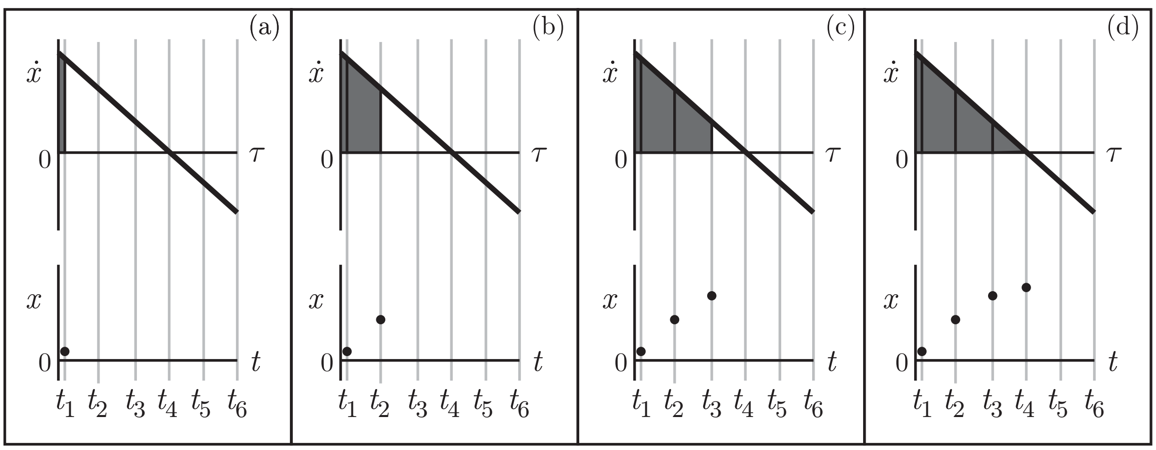

Figure B.8: Integration as the area under a curve.

So, using expression (B.2), we can write $x(t_1)$ as

$$x(t_1) = \int_0^{t_1} \dot{x}(\tau)\, d\tau.$$

Notice from the figure that t_1 is very close to zero. Therefore, the integral above represents the area in a very narrow slice as shown in Figure B.8(a). The area is almost nothing. Therefore, the value of $x(t_1)$ is almost zero as indicated in the bottom half of the sub-figure.

If we were to substitute $t = t_2$ into (B.2), then we would find that $x(t_2)$ is the area under the $\dot{x}(\tau)$ curve between $\tau = 0$ and $\tau = t_2$. Since the area between 0 and t_2 is larger than the area between 0 and t_1, we know that $x(t_2) > x(t_1)$, as indicated in the figure.

As we let t increase to t_3 and to t_4, the integral keeps *accumulating* more area. Therefore,

$$x(t_2) < x(t_3) < x(t_4).$$

[3]You remember integration by parts, right?

The function $x(t)$ continues to increase between 0 and t_4.

However, notice that the area accumulated between t_3 and t_4 is not as large as the area accumulated between t_2 and t_3. For this reason, $x(t)$ does not increase as much between t_3 and t_4 as it did between t_2 and t_3.

In Figure B.9, I continue the illustration begun in Figure B.8. Between t_4 and t_5, $\dot{x}(\tau)$ is negative. Therefore, the area circumscribed during this interval makes a negative contribution to the integral. Therefore, $x(t_5)$ is less than $x(t_4)$. In particular, since $\dot{x}(\tau)$ is a straight line and all time intervals have the same width, the positive area between t_3 and t_4 has the same magnitude as the negative area between t_4 and t_5. Thus, the increment in $x(t)$ between t_3 and t_4 is equal to the decrement between t_4 and t_5. As a consequence, $x(t_3) = x(t_5)$.

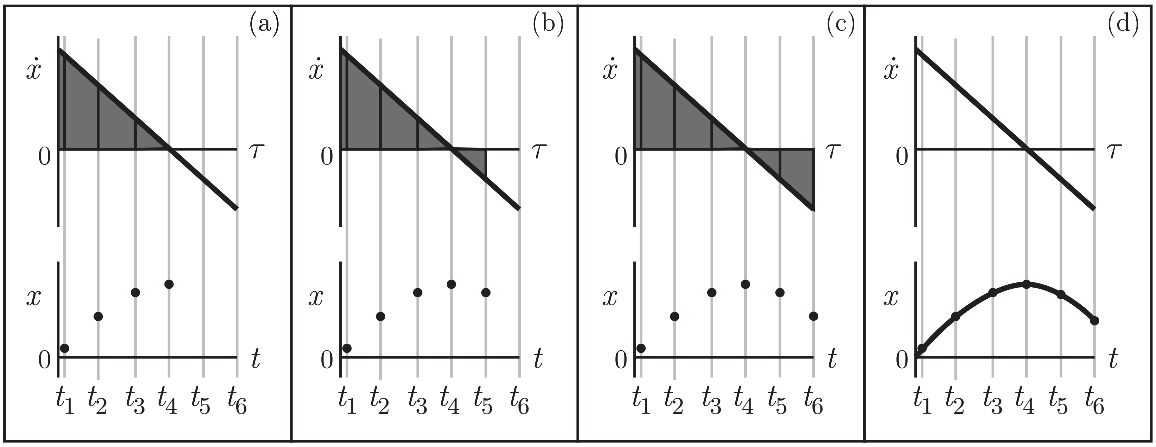

Figure B.9: Continued discussion of integration as the area under a curve.

Similarly, in going from t_5 to t_6, the accumulated area is more negative, and $x(t)$ decreases more rapidly. Because of the symmetry discussed in the previous paragraph, we have $x(t_2) = x(t_6)$.

Of course, $x(t)$ is a continuous function defined at more than just the six instances of times highlighted in our discussion so far. By connecting the dots as shown in Figure B.9 we get a parabolic-looking[4] curve for $x(t)$.

B.2.5 Our Two Interpretations of the Integral are Consistent

So in the previous several pages we've seen integration being interpreted as an anti-derivative. Then we saw integration interpreted as the area under a curve. Although these may seem like completely different interpretations, they are entirely consistent.

When $\dot{x}(t)$ is more positive, our first interpretation as an anti-derivative tells us that that $x(t)$ should increase more rapidly. Similarly, our area interpretation tells us that when $\dot{x}(t)$ (integrand) is more positive, the integral accumulates area at a faster rate. Therefore, $x(t)$ increases at a faster rate.

Furthermore, note that our function $\dot{x}(\tau)$ in Figures B.9(d) is the same as the first part of $\dot{x}(t)$ in Figure B.6. Also notice that our integrals $x(t)$ obtained via the two interpretations are the same.

Using our area interpretation, we can explain a few features of $x(t)$ in Figure B.6 that were difficult to understand just by thinking of the integral as an anti-derivative.

[4]Recall that the integral of a linear function is quadratic. Therefore, $x(t)$ is, indeed, parabolic.

For example, $x(t_2)$ in Figure B.6 is positive because the positive area under the $\dot{x}(t)$ between 0 and t_1 is bigger in magnitude than the negative area "under" $\dot{x}(t)$ between t_1 and t_2.

Similarly $x(t_5)$ is negative because the negative area between t_2 and t_3 has larger magnitude than the positive area after t_3.

B.2.6 Take-Aways

Here is a list of what I feel are the most important points about integration that you should take away from Appendix B.2

1. We discussed two ways of thinking about integration:
   (a) An integral is an "anti-derivative," the opposite of derivative.
   (b) An integral can also be interpreted as the area "under" a curve.
2. When the integrand (the function you're integrating) is *positive*, the integral is *increasing.*
3. When the integrand is *negative*, the integral is *decreasing.*
4. When the integrand is *zero*, the integral *does not change.*
5. For indefinite integrals, there is an arbitrary constant of integration.