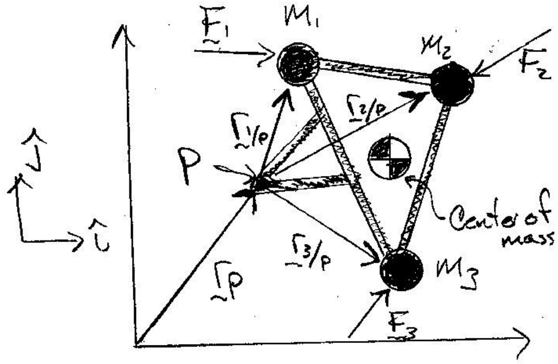


Planar dynamics of a rigid body

Suppose we have a rigid body consisting of 3 particles connected by massless rods.

Suppose, also, that we choose some point P connected to the rigid body. The point P, so far, is arbitrary.

Note that this is a collection of particles, and we know from a previous theory video that only the external forces matter in the dynamics of this object. In particular, we know

$$\boxed{\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = m_{TOT}\, \underset{\sim}{a}_G}$$

In these notes, we'll set up the moment equations for the three-particle body

Note that $\underset{\sim}{r}_1 = \underset{\sim}{r}_p + \underset{\sim}{r}_{1/p}$

$$\underset{\sim}{r}_2 = \underset{\sim}{r}_p + \underset{\sim}{r}_{2/p}$$

$$\underset{\sim}{r}_3 = \underset{\sim}{r}_p + \underset{\sim}{r}_{3/p}$$

Let's start w/ the dynamics of particle 1

$$\underset{\sim}{F}_1 = m_1 \underset{\sim}{a}_1 = m_1\left(\underset{\sim}{a}_p + \underset{\sim}{a}_{1/p}\right)$$

$$\underset{\sim}{F}_1 = m_1\left(\underset{\sim}{a}_p + \underset{\sim}{\alpha} \times \underset{\sim}{r}_{1/p} + \underset{\sim}{\omega} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{r}_{1/p}\right)\right)$$

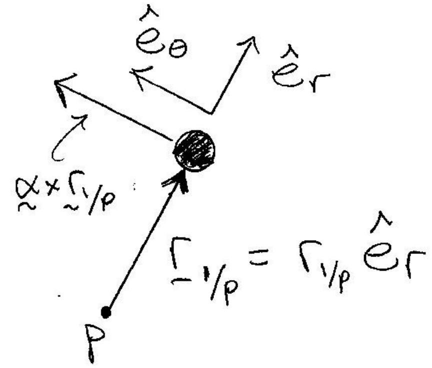
where $\underset{\sim}{\omega} = \omega\hat{k}$ is the angular velocity &
$\underset{\sim}{\alpha} = \alpha\hat{k}$ is the angular acceleration of the body

Now we'll take a "moment" about point P.

$$\underset{\sim}{r}_{1/p} \times \underset{\sim}{F}_1 = \underset{\sim}{r}_{1/p} \times m_1\left[\underset{\sim}{a}_p + \underset{\sim}{\alpha} \times \underset{\sim}{r}_{1/p} + \underset{\sim}{\omega} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{r}_{1/p}\right)\right] \quad (1)$$

$$\underset{\sim}{M}_{1/p} = \underbrace{\underset{\sim}{r}_{1/p} \times m_1 \underset{\sim}{a}_p}_{\text{first term}} + \underbrace{\underset{\sim}{r}_{1/p} \times m_1\left(\underset{\sim}{\alpha} \times \underset{\sim}{r}_{1/p}\right)}_{\text{Second term}} + \underbrace{\underset{\sim}{r}_{1/p} \times m_1\left(\underset{\sim}{\omega} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{r}_{1/p}\right)\right)}_{\text{Third term}}$$

The second & third terms are rather messy, so let's tackle them individually



Observe that

$$\underset{\sim}{\alpha} \times \underset{\sim}{r}_{1/p} = \alpha \hat{k} \times r_{1/p} \hat{e}_r$$
$$= \alpha r_{1/p} \hat{e}_\theta$$

So

$$\underset{\sim}{r}_{1/p} \times (\underset{\sim}{\alpha} \times \underset{\sim}{r}_{1/p}) = r_{1/p} \hat{e}_r \times \alpha r_{1/p} \hat{e}_\theta$$
$$= \alpha r_{1/p}^2 \hat{k} \quad (2)$$

Next, the third term

$$\underset{\sim}{r}_{1/p} \times (\underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}_{1/p})) = r_{1/p} \hat{e}_r \times (\omega \hat{k} \times (\omega \hat{k} \times r_{1/p} \hat{e}_r))$$
$$= r_{1/p} \hat{e}_r \times (\omega \hat{k} \times \omega r_{1/p} \hat{e}_\theta)$$
$$= r_{1/p} \hat{e}_r \times (-\omega^2 r_{1/p} \hat{e}_r)$$
$$= 0 \quad (3)$$

I should tell you that this term is not necessarily zero for 3-D rotations .... But for planar dynamics, it simplifies things

Substituting (2) & (3) into (1), we get

$$\underset{\sim}{M}_{1/p} = m_1 \underset{\sim}{r}_{1/p} \times \underset{\sim}{a}_p + m_1 \alpha r_{1/p}^2 \hat{k} \quad (4)$$

If we do the exact same thing for the other two particles we get

$$\underset{\sim}{M}_{2/p} = m_2 \underset{\sim}{r}_{2/p} \times \underset{\sim}{a}_p + m_2 \alpha r_{2/p}^2 \hat{k} \qquad (4b)$$

$$\underset{\sim}{M}_{3/p} = m_3 \underset{\sim}{r}_{3/p} \times \underset{\sim}{a}_p + m_3 \alpha r_{3/p}^2 \hat{k} \qquad (4c)$$

Now let's add the three equations together

$$\underset{\sim}{M}_{1/p} + \underset{\sim}{M}_{2/p} + \underset{\sim}{M}_{3/p} = \left(m_1 \underset{\sim}{r}_{1/p} + m_2 \underset{\sim}{r}_{2/p} + m_3 \underset{\sim}{r}_{3/p}\right) \times \underset{\sim}{a}_p$$
$$+ \left(m_1 r_{1/p}^2 + m_2 r_{2/p}^2 + m_3 r_{3/p}^2\right) \alpha \hat{k} \qquad (5)$$

Let's look at the stuff in the first set of parentheses

Recall

$$\left(m_1 \underset{\sim}{r}_{1/p} + m_2 \underset{\sim}{r}_{2/p} + m_3 \underset{\sim}{r}_{3/p}\right) = m_{Tot} \, \underset{\sim}{r}_{G/p}$$

total mass

position of center of mass relative to P

The sum of terms we see in the second set of parentheses is something we see in mechanics alot. It is called the moment of inertia about P

$$I_P = m_1 r_{1/p}^2 + m_2 r_{2/p}^2 + m_3 r_{3/p}^2$$

Therefore, we can write (5) as

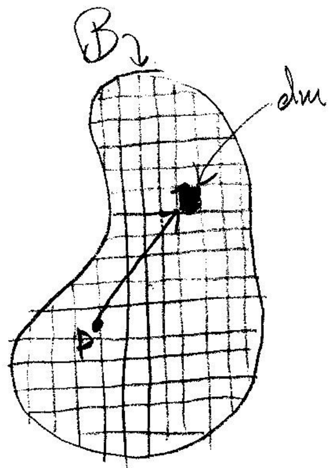
$$\boxed{\underset{\sim}{M}_P^{total} = \underset{\sim}{r}_{G/P} \times m_{tot}\, \underset{\sim}{a}_P + I_P\, \alpha\, \hat{k}} \quad (*)$$

This is an important & powerful equation. Although we derived it for a rigid body comprised of three particles, one can generalize it to any number of particles

where $$\underset{\sim}{r}_{G/P} = \frac{\sum_{j=1}^{N} m_j\, \underset{\sim}{r}_{j/P}}{\sum_{j=1}^{N} m_j}$$

$$\& \quad I_P = \sum_{j=1}^{N} m_j\, r_{j/P}^2$$

← Notice that this has units of (ML^2) mass times length squared



$$\underset{\sim}{r}_{G/P} = \iint_B r_{dm/P}\, dm \Big/ \iint_B dm$$

$$I_P = \iint_B r_{dm/P}^2\, dm$$