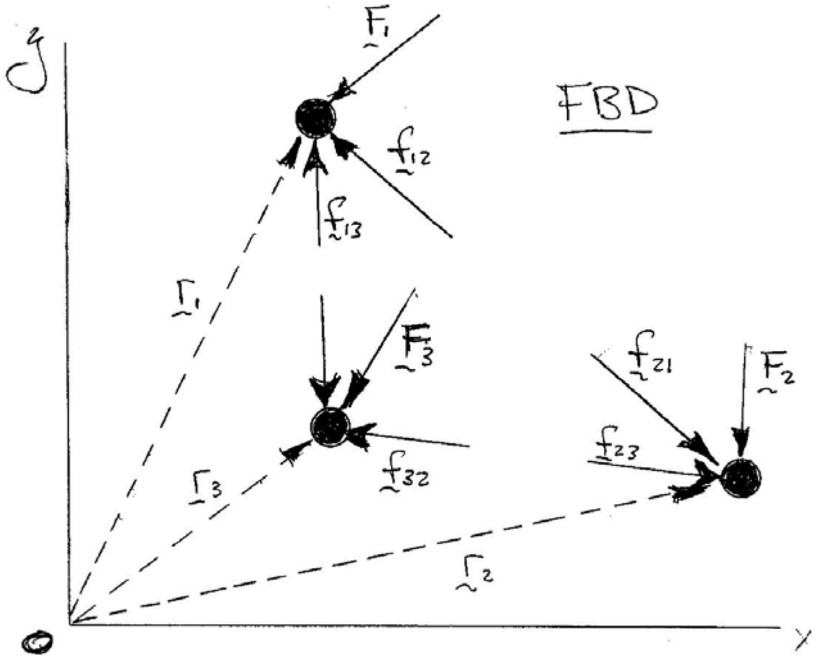
Dynamics of a collection of particles, Part 2



Just like in the first part of these notes, let's suppose we have three particles. As before, we let $\underset{\sim}{F}_1, \underset{\sim}{F}_2, \underset{\sim}{F}_3$ denote the external forces acting on the three particles

Let $\underset{\sim}{f}_{ij}$ denote the internal forces between the particles

Also shown (dashed) are the positions of the particles relative to the point O. ← A point that we define

In those previous notes we applied Newtons 2nd law to arrive at the following equations of motion

Particle 1: $\underset{\sim}{F}_1 + \underset{\sim}{f}_{12} + \underset{\sim}{f}_{13} = m_1 \underset{\sim}{a}_1$

Particle 2: $\underset{\sim}{F}_2 + \underset{\sim}{f}_{21} + \underset{\sim}{f}_{23} = m_2 \underset{\sim}{a}_2$

Particle 3: $\underset{\sim}{F}_3 + \underset{\sim}{f}_{31} + \underset{\sim}{f}_{32} = m_3 \underset{\sim}{a}_3$

For each particle, we take the moment about point O

$$\left.\begin{aligned}
&\text{Particle 1:} \quad \underset{\sim}{r}_1 \times \left(\underset{\sim}{F}_1 + \underset{\sim}{f}_{12} + \underset{\sim}{f}_{13}\right) = \underset{\sim}{r}_1 \times m_1 \underset{\sim}{a}_1 \\
&\text{Particle 2:} \quad \underset{\sim}{r}_2 \times \left(\underset{\sim}{F}_2 + \underset{\sim}{f}_{21} + \underset{\sim}{f}_{23}\right) = \underset{\sim}{r}_2 \times m_2 \underset{\sim}{a}_2 \\
&\text{Particle 3} \quad \underset{\sim}{r}_3 \times \left(\underset{\sim}{F}_3 + \underset{\sim}{f}_{31} + \underset{\sim}{f}_{32}\right) = \underset{\sim}{r}_3 \times m_3 \underset{\sim}{a}_3
\end{aligned}\right\} \quad (*)$$

As before, the internal forces come in equal & opposite pairs (Newton's 3rd law)

$$\underset{\sim}{f}_{21} = -\underset{\sim}{f}_{12} \; ; \; \underset{\sim}{f}_{31} = -\underset{\sim}{f}_{13} \; ; \; \underset{\sim}{f}_{32} = -\underset{\sim}{f}_{23}$$

Substituting these into (*) & distributing the cross product, we get.

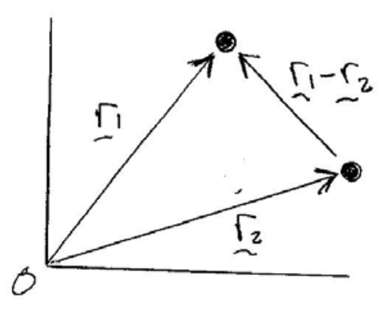
$$\begin{cases} \underset{\sim}{r}_1 \times \underset{\sim}{F}_1 + \underset{\sim}{r}_1 \times \underset{\sim}{f}_{12} + \underset{\sim}{r}_1 \times \underset{\sim}{f}_{13} = \underset{\sim}{r}_1 \times m_1 \underset{\sim}{a}_1 \\ \underset{\sim}{r}_2 \times \underset{\sim}{F}_2 + \underset{\sim}{r}_2 \times (-\underset{\sim}{f}_{12}) + \underset{\sim}{r}_2 \times \underset{\sim}{f}_{23} = \underset{\sim}{r}_2 \times m_2 \underset{\sim}{a}_2 \\ \underset{\sim}{r}_3 \times \underset{\sim}{F}_3 + \underset{\sim}{r}_3 \times (-\underset{\sim}{f}_{13}) + \underset{\sim}{r}_3 \times (-\underset{\sim}{f}_{23}) = \underset{\sim}{r}_3 \times m_3 \underset{\sim}{a}_3 \end{cases}$$

Now we add up the equations, we get

$$\left.\begin{aligned} &\underset{\sim}{r}_1 \times \underset{\sim}{F}_1 + \underset{\sim}{r}_2 \times \underset{\sim}{F}_2 + \underset{\sim}{r}_3 \times \underset{\sim}{F}_3 + (\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \times \underset{\sim}{f}_{12} \\ &\quad + (\underset{\sim}{r}_2 - \underset{\sim}{r}_3) \times \underset{\sim}{f}_{23} + (\underset{\sim}{r}_1 - \underset{\sim}{r}_3) \times \underset{\sim}{f}_{13} = \\ &\qquad \underset{\sim}{r}_1 \times m_1 \underset{\sim}{a}_1 + \underset{\sim}{r}_2 \times m_2 \underset{\sim}{a}_2 + \underset{\sim}{r}_3 \times m_3 \underset{\sim}{a}_3 \end{aligned}\right\} (\#)$$

To obtain the equation, we combined terms of like internal force

In looking at particles 1 & 2, observe that the vector $\underset{\sim}{r}_1 - \underset{\sim}{r}_2$ is parallel to $\underset{\sim}{f}_{12}$.



Therefore the term $(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \times \underset{\sim}{f}_{12}$ must be ZERO.

Similarly one can show that

$$(\underset{\sim}{r}_2 - \underset{\sim}{r}_3) \times \underset{\sim}{f}_{23} = 0 \;\; \& \;\; (\underset{\sim}{r}_1 - \underset{\sim}{r}_3) \times \underset{\sim}{f}_{13} = 0$$

Therefore, all the internal forces go away again.

Note that $\underset{\sim}{r}_1 \times \underset{\sim}{F}_1 = \underset{\sim}{M}_{1_0}$; $\underset{\sim}{r}_2 \times \underset{\sim}{F}_2 = M_{2_0}$;

$\underset{\sim}{r}_3 \times \underset{\sim}{F}_3 = \underset{\sim}{M}_{3_0}$

are the external moments about point O

Then Equation (#) becomes

$$\underbrace{\underset{\sim}{M}_{1_0} + \underset{\sim}{M}_{2_0} + \underset{\sim}{M}_{3_0}}_{\text{This is the total external moment about point O}} = \underbrace{\underset{\sim}{r}_1 \times m_1 \underset{\sim}{a}_1 + \underset{\sim}{r}_2 \times m_2 \underset{\sim}{a}_2 + \underset{\sim}{r}_3 \times m_3 \underset{\sim}{a}_3}_{\text{What is this?}} \qquad (\blacktriangle)$$

Note that the angular momentum about point O is

$$\underset{\sim}{H}_o^{total} = \underset{\sim}{r}_1 \times m_1 \underset{\sim}{v}_1 + \underset{\sim}{r}_2 \times m_2 \underset{\sim}{v}_2 + \underset{\sim}{r}_3 \times m_3 \underset{\sim}{v}_3$$

Taking a time derivative, we get

$$\dot{\underset{\sim}{H}}_o^{total} = \dot{\underset{\sim}{r}}_1 \times m_1 \underset{\sim}{v}_1 + \underset{\sim}{r}_1 \times m_1 \dot{\underset{\sim}{v}}_1 + \dot{\underset{\sim}{r}}_2 \times m_2 \underset{\sim}{v}_2 + \underset{\sim}{r}_2 \times m_2 \dot{\underset{\sim}{v}}_2 + \dot{\underset{\sim}{r}}_3 \times m_3 \underset{\sim}{v}_3 + \underset{\sim}{r}_3 \times m_3 \dot{\underset{\sim}{v}}_3$$

Note that $\dot{\underset{\sim}{r}}_1 = \underset{\sim}{v}_1$ by definition

$\underset{\sim}{b} \times \underset{\sim}{b} = 0$ for any vector $\underset{\sim}{b}$

$$\Rightarrow \dot{\underset{\sim}{r}}_1 \times m_1 \underset{\sim}{v}_1 = \underset{\sim}{v}_1 \times m_1 \underset{\sim}{v}_1 = 0$$

Similarly, $\dot{\underset{\sim}{r}}_2 \times m \underset{\sim}{v}_2 = \underset{\sim}{v}_2 \times m_2 v_2 = 0$

$\dot{\underset{\sim}{r}}_3 \times m \underset{\sim}{v}_3 = \underset{\sim}{v}_3 \times m_3 \underset{\sim}{v}_3 = 0$

So,

$$\dot{\underset{\sim}{H}}_o^{total} = \underset{\sim}{r}_1 \times m_1 \dot{\underset{\sim}{v}}_1 + \underset{\sim}{r}_2 \times m_2 \dot{\underset{\sim}{v}}_2 + \underset{\sim}{r}_3 \times m_3 \dot{\underset{\sim}{v}}_3$$

$$= \underset{\sim}{r}_1 \times m_1 \underset{\sim}{a}_1 + \underset{\sim}{r}_2 \times m_2 \underset{\sim}{a}_2 + \underset{\sim}{r}_3 \times m_3 \underset{\sim}{a}_3$$

Therefore Equation (▲) becomes

$$\boxed{\underset{\sim}{M}_o^{total} = \dot{\underset{\sim}{H}}_o^{total}}$$

Time derivative of angular momentum about O for all particles

Total external moment about O for all particles

This result is true for any number of particles & also in three dimensions.