Appendix A

Review of Vectors

Almost everything we do in engineering dynamics is based on Newton's second law:

$$\sum \vec{F} = m\,\vec{a} \tag{A.1}$$

It is important to note that this is a *vector* equation. On the left side of the equation is the sum of all force vectors acting on an object. On the right side is the vector corresponding to the acceleration of the object. To make sense of this, and to use it properly, we are going to have to be able to work with vectors and understand how to manipulate them mathematically.

A.1 Representation of Vectors

In this class, we'll generally think of vectors as quantities that have both magnitude and direction. When you push on something, you can push it hard or gently (magnitude). Furthermore, you can push it to the left, to the right or in any direction. We'll represent that push (force) as a vector.

It is important to note that there is one special vector that does not have a direction. It is the *zero vector*. In the context of forces, think of the zero vector as not pushing or pulling in any direction.

A.1.1 Graphical Representation

When thinking about how vectors add, subtract, or multiply, it is often best to think of vectors graphically as an arrow.

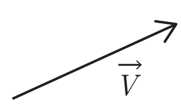


Figure A.1: Graphical representation of vector.

The arrow points in the direction of the vector. We represent the magnitude of the vector by the length of the arrow. A ten pound force vector would be twice as long, graphically, as a five pound force vector.

A.1.2 Symbolic Representation

Of course, we are going to need to express vectors symbolically as well. Equation (A.1) is such an example. Because vectors *are* different from simple numbers (scalars), we *write* them differently. So far in this Appendix, I have indicated vectors as symbols with small arrows above: $\vec{F} = m\vec{a}$.

You probably noticed the little squiggly lines underneath the symbols to indicate the vectors above. Books and scientific articles often denote vectors by boldface type like: $\boldsymbol{F} = m\boldsymbol{a}$. Sometimes, you might see me write vectors as symbols with "squiggles" underneath like this: $\underset{\sim}{F} = m\,\underset{\sim}{a}$. I was taught this second way, and secretly I prefer it. But I am trying to change so that my notation is consistent with the majority of people who teach and study mechanics.

Personally, I do not care which notation you choose to use for vectors. However, *it is important that you write vectors in a way that distinguishes them from scalars.* This is important because the arithmetic (addition, subtraction, multiplication, and division(?) for vectors is different than that of scalars. Physically, they have different meaning as well.

In addition, it is common to write unit vectors (vectors with magnitude 1.0) with "hats" on them: $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, $\hat{e}$. You may do this as well. However it's OK if you use the general vector notation for unit vectors.

A.2 Vector Arithmetic I: Multiplying a Vector by a Scalar

Back in elementary school, you learned how to do arithmetic with scalars: add, subtract, multiply, and divide. Since vectors are different than scalars, it makes sense that vector arithmetic is different than scalar arithmetic. There are two different ways of multiplying a vector by another vector. You can't divide anything by a vector.

Hopefully, you have already learned vector arithmetic. In this appendix, we review it in Sections A.2, A.3, A.5, and A.6 for your reference. We start with the simplest case.

Mathematically, one of the most fundamental things you can do with a vector is multiply it by a scalar. Suppose we have a vector $\vec{V}$. If I take $\vec{V}$ and multiply it by the scalar 3, then I get another vector which I call $3\,\vec{V}$. (see Figure A.2.) Both $\vec{V}$ and $3\,\vec{V}$ have the same direction. However the magnitude of $3\,\vec{V}$ is three times as large as $\vec{V}$. Simple, eh?

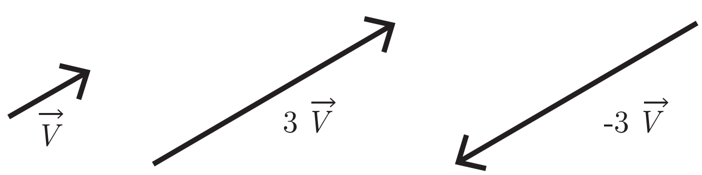


Figure A.2: Scalar multiplication.

Observe that if $\hat{\imath}$ is a unit vector (has magnitude one), then for any scalar c, the vector $c\,\hat{\imath}$ has magnitude $|c|$. If c is positive, then both $c\,\hat{\imath}$ and $\hat{\imath}$ point in the same direction. If c is negative, then $c\,\hat{\imath}$ and $\hat{\imath}$ point in opposite directions. And if $c = 0$, then $c\,\hat{\imath}$ is the zero vector (which has zero magnitude, and its direction isn't well defined).

A.3 Vector Arithmetic II: Adding Two Vectors

Suppose that you're pushing on the car below with a force of 100 lb to the right (force has magnitude and direction). Also suppose that your friend is pulling on the car with a force of 75 lb in the same direction. (Your friend is not as strong as you.) The two of you together produce a total force of 175 lb to the right. In other words, the two force vectors add together.

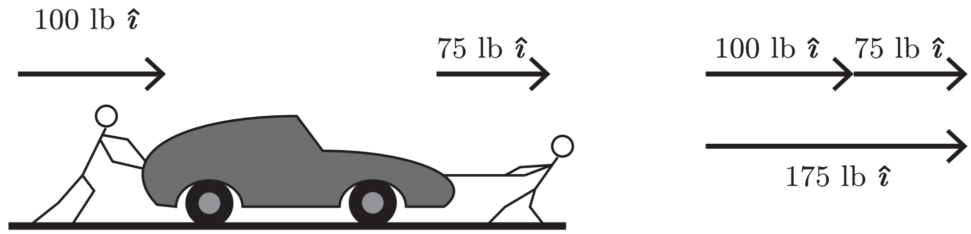


Figure A.3: When two people push/pull on an object, their force vectors add together.

A.3.1 Thinking About Vector Addition Graphically

Graphically, we think of vectors as arrows. The direction of the arrow represents the direction of the vector, and the length of the arrow represents the magnitude. If we take our two force vectors from the car pushing/pulling example of Figure A.3, and lay them end to end (head to tail), then we can think of the total force vector as the vector that starts at the first tail and ends at the last head: a vector pointing to the right with magnitude 175 lb.

This is how we will think of vector addition more generally. If we have two vectors $\vec{P}$ and $\vec{Q}$, then $\vec{P}+\vec{Q}$ can be obtained by placing the two vectors arrows head-to-tail as shown in Figure A.4. The sum is the vector (arrow) that starts at the tail of the first vector and ends at the head of the second. Notice that the head-to-tail ordering of the vectors doesn't matter, so $\vec{P}+\vec{Q}=\vec{Q}+\vec{P}$.[1]

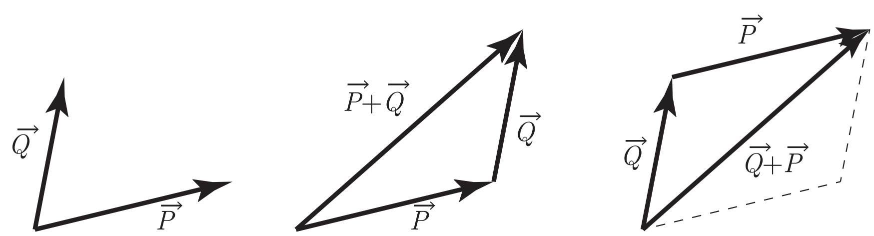


Figure A.4: Forming the sum $\vec{P}+\vec{Q}$ and $\vec{Q}+\vec{P}$ by arranging the vectors $\vec{P}$ and $\vec{Q}$ in head-to-tail fashion.

A.3.2 What About Vector Subtraction?

Check this out. Suppose we have vectors $\vec{P}$ and $\vec{Q}$ from the previous section, and we define a new vector $\vec{R}$ which connects the two tips as shown in Figure A.5. Is there another way we can think about vector $\vec{R}$?

Observe from the figure that, according to our graphical definition of a vector sum (Section A.3.1), we know

$$\vec{Q}=\vec{P}+\vec{R}.$$

Manipulating this equation just slightly, we obtain

$$\vec{R}=\vec{Q}-\vec{P}. \tag{A.2}$$

[1]We say that vector addition is *commutative*.

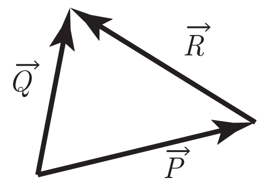


Figure A.5:

This tells us what vector subtraction means. The difference between vectors $\vec{Q}$ and $\vec{P}$ is a vector that connects the tips of the two vectors, originating at the tip of $\vec{P}$ and directed toward the tip of vector $\vec{Q}$.

A.3.3 First Look at Vector Decomposition: Basis Vectors

Suppose we have some vector $\vec{V}$ and two other unit vectors $\hat{\imath}$ and $\hat{\jmath}$ which are perpendicular to each other. Suppose that all three of these vectors lie in a plane as depicted in Figure A.6a.

Using the idea of vector addition, it is possible to write vector $\vec{V}$ in terms of vectors $\hat{\imath}$ and $\hat{\jmath}$. The idea is presented graphically in Figure A.6b. We draw one vector parallel to $\hat{\imath}$ and another vector parallel to $\hat{\jmath}$ such that when we arrange them in head-to-tail order, they "add up" to vector $\vec{V}$.

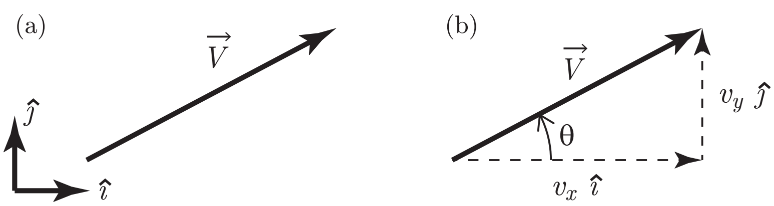


Figure A.6: Illustration showing how a vector $\vec{V}$ can be decomposed into components in the $\hat{\imath}$ and $\hat{\jmath}$ directions.

If we let v_x (a scalar) denote the magnitude (length) of the component vector in the $\hat{\imath}$ direction, and v_y denote the magnitude of the component vector in the $\hat{\jmath}$ direction, then we can write

$$\vec{V} = v_x\,\hat{\imath} + v_y\,\hat{\jmath}. \tag{A.3}$$

The equation above is called the *vector decomposition* of $\vec{V}$. Quantities v_x and v_y are called the *components* of vector $\vec{V}$ in the $\hat{\imath}$ and $\hat{\jmath}$ directions respectively. Furthermore, $\hat{\imath}$ and $\hat{\jmath}$ are called the *basis vectors* for the decomposition.

Suppose that we know the magnitude of vector $\vec{V}$; we'll call it v. Suppose, also, that we know that the direction of vector $\vec{V}$ is oriented at an angle θ relative to the $\hat{\imath}$ basis vector. Then, writing the components v_x and v_y is simple trigonometry:

$$v_x = v\,\cos(\theta); \qquad v_y = v\,\sin(\theta). \tag{A.4}$$

It's worth noting that the reason why the formulas in (A.4) are so simple is because the triangle in Figure A.6b is a right triangle. And the reason why it is a right triangle is because the basis vectors $\hat{\imath}$ and $\hat{\jmath}$ are perpendicular. In Engineering Dynamics, we'll have good reasons to choose different types of basis vectors, but we'll always choose basis vectors that are perpendicular so that the math works out a little easier.

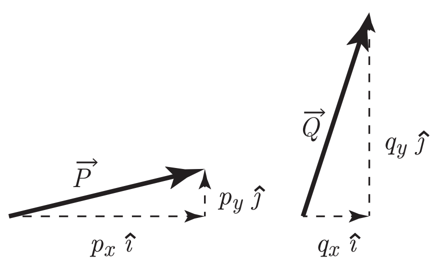
Given that so many students have difficulty with vector decomposition when it comes to solving problems, I have written a separate section (Section A.4) which covers some of the practical aspects of the subject.

A.3.4 Thinking About Vector Addition Symbolically

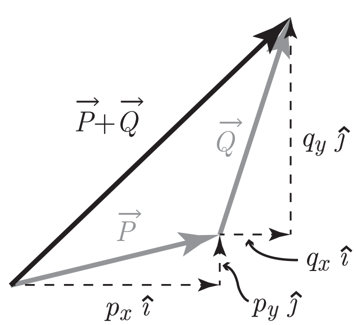
Suppose we have two vector $\vec{P}$ and $\vec{Q}$ which we can express symbolically in terms of their components:

$$\vec{P} = p_x\,\hat{\imath} + p_y\,\hat{\jmath}; \qquad \vec{Q} = q_x\,\hat{\imath} + q_y\,\hat{\jmath}.$$

This is equivalent to what we have depicted graphically in the figure below.



Next, in the figure below, we add the two vectors together.



Symbolically, this is equivalent to

$$\vec{P} + \vec{Q} = (p_x\,\hat{\imath} + p_y\,\hat{\jmath}) + (q_x\,\hat{\imath} + q_y\,\hat{\jmath}).$$

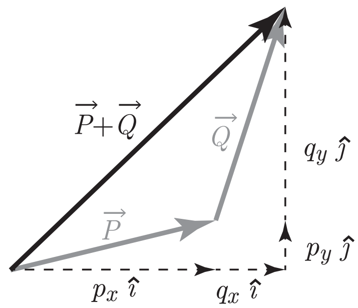
Now, in the diagram below, we show graphically that we can re-arrange components in the sum. The symbolic equivalent is

$$\vec{P} + \vec{Q} = (p_x\,\hat{\imath} + q_x\,\hat{\imath}) + (p_y\,\hat{\jmath} + q_y\,\hat{\jmath}).$$

Making one more minor simplification, we can write

$$\vec{P} + \vec{Q} = (p_x + q_x)\,\hat{\imath} + (p_y + q_y)\,\hat{\jmath}. \tag{A.5}$$

Thus, the algebra of adding two vectors is quite straightforward. Vectors can be added componentwise.



A.4 More on Vector Decomposition (Video1) (Video2)

Back in your Engineering Statics class (or earlier), you should have become a pro at vector decomposition. However, it has been my experience that many students still struggle with it. Therefore, in this section we are going to go over an example, in detail. The example covers the main elements of vector decomposition that I expect you to know and that I expect you to perform easily and almost automatically.

If you prefer to watch and listen, rather than read, I have posted two video which cover the content of this section: (Video1) (Video2). Before reading (or watching) this section, you should have read Section A.3.3. We will build off of that material.

As a concrete example with which to discuss vector decomposition, let's reconsider the problem of a block sliding down a ramp. In Figure A.7, we draw the problem as well as a free body diagram depicting the force vectors acting on the block.

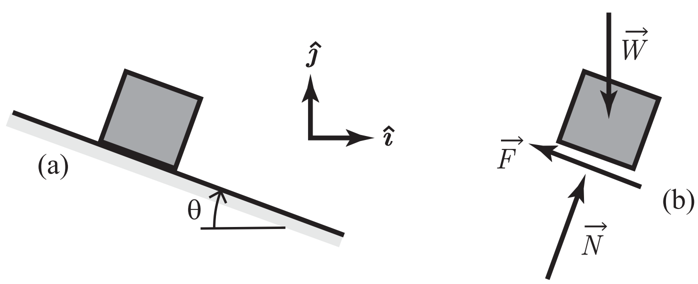


Figure A.7: Problem of a block sliding down a ramp, with associated free body diagram.

The vector $\vec{W}$ represents the force of gravity. Since gravity always pulls downward, the weight vector is an arrow that points down. Vector $\vec{N}$ is the normal force from the ramp acting on the block. It is the force that prevents the block from falling through the table; it acts perpendicular to the ramp. Therefore, we see the $\vec{N}$ vector pointed in a direction perpendicular to the ramp. The final force $\vec{F}$ is due to friction. It opposes the motion of the block as it slides down. Thus $\vec{F}$ points in a direction parallel to the ramp, upward and to the left. Our task is to take the three vectors, $\vec{W}$, $\vec{F}$, and $\vec{N}$, and write them in terms of basis vectors $\hat{\imath}$ and $\hat{\jmath}$.

The easiest of the three is the weight, $\vec{W}$, since it acts in the direction of $\hat{\jmath}$ (actually $-\hat{\jmath}$). We can write

$$\boxed{\vec{W} = -mg\,\hat{\jmath}.}$$

Here, mg is the magnitude of the weight, both m and g are positive.

That was trivial. The vectors $\vec{F}$ and $\vec{N}$ will be a little more difficult. To decompose a vector, any vector, into components, there is a general recipe that one can follow. I list the steps below and then demonstrate how they are applied to the friction and normal forces in our example problem.

A.4.1 A Recipe for Vector Decomposition

Here it is:

1. Draw a force triangle.
   (a) Two sides of the triangle should be parallel to the basis vectors (e.g. the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ vectors.)
   (b) The third side of the triangle (the hypotenuse) should be the vector you want to decompose.
   (c) Put appropriate arrows on each side of the force triangle. The arrow on the hypotenuse should point in the direction of the original vector that we want to decompose. The other two vectors (component vectors) should be aligned head-to-tail so so that they add up to the hypotenuse vector.
2. Identify a key angle [or side ratio].
   (a) Because the basis vectors are perpendicular to each other, one angle of the triangle is 90°. It is a right triangle. Therefore, if we can determine one of the other angles of the triangle, then we can calculate anything we want to know about the triangle.
   (b) Sometimes, it's easier to get the ratio of two sides of the triangle rather than an angle. I'll show you an example of this in Section A.4.5
3. Use trigonometric principles to calculate the magnitudes of the components vectors
   (a) I know that some of you have mnemonics (e.g. *SOH-CAH-TOA* or "**S**ome **O**ld **H**ippy **C**aught **A**nother **H**ippy **T**rippin' **O**n **A**cid) to remember the definitions of the elementary trigonometric functions. Now that you're an engineer, it's time for you to lose the crutch. Commit the definitions to memory!
4. Write the original vector in terms of its components.
   (a) Be sure to get the signs of the components right. If the component vector from step 1(c) points in the same direction as the basis vector, then the component should be positive. If they point in the opposite direction, the component should be negative.
5. Check that your answer makes sense.

There are other "recipes" for decomposing a vector. I find this one to be convenient and relatively easy to understand.

A.4.2 Applying the Recipe to the Friction Force

In decomposing the friction vector $\vec{F}$ of Figure A.7, let's say the magnitude of $\vec{F}$ is F.

Step 1. For friction, the vector triangle is shown in Figure A.8. Component vectors $F_x\,\hat{\imath}$ and $F_y\,\hat{\jmath}$ are the vectors discussed in Step 1a. As you can see, the friction vector itself is the hypotenuse of the triangle as specified in Step 1b. Notice also that the arrows on the component vectors point in a direction so that $F_x\,\hat{\imath}$ and $F_y\,\hat{\jmath}$ add up to $\vec{F}$ as described in Step 1c.

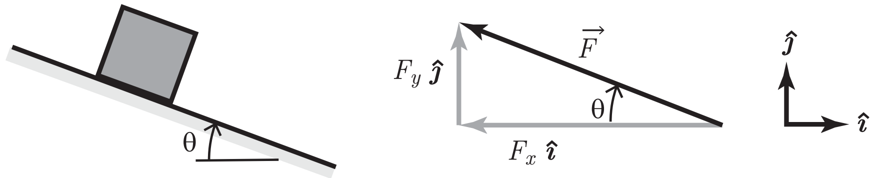


Figure A.8: Decomposition of the friction vector.

Step 2. In our example problem, the friction vector is parallel to the ramp, and the ramp is at an angle θ relative to the horizontal. Therefore, the angle in the bottom right corner of the vector triangle is θ.

Step 3. Now it's time for trigonometry. Here we get $|F_x| = F\,\cos(\theta)$, $|F_y| = F\,\sin(\theta)$.

Step 4. Here, we put the pieces together to write the vector decomposition:

$$\boxed{\vec{F} = -F\,\cos(\theta)\,\hat{\imath} + F\,\sin(\theta)\,\hat{\jmath}.} \tag{A.6}$$

Notice that the $\hat{\imath}$ component of $\vec{F}$ is negative. This is because component vector $F_x\,\hat{\imath}$ in Figure A.8 points in the opposite direction as the basis vector $\hat{\imath}$. The $\hat{\jmath}$ component is positive since since the vertical component and basis vector point in the same direction.

Step 5. We'll postpone checking our answer until Section A.4.3.

A.4.3 Applying the Recipe to the Normal Force

Let's do it again, but this time for the normal force.

Step 1. In Figure A.9, I show the vector triangle for the Normal force. Again, the vector we want to decompose is the hypotenuse of the triangle. The other two sides are parallel to the $\hat{\imath}$ and $\hat{\jmath}$ directions, with arrows pointing in directions so that $\vec{N} = N_x\,\hat{\imath} + N_y\,\hat{\jmath}$.

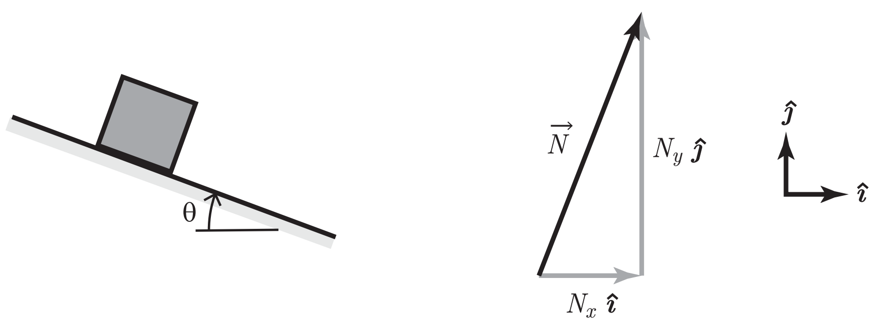


Figure A.9: Force triangle for the normal vector.

Step 2. Identifying a key angle for the force triangle is a little trickier for the normal vector. There are several ways of doing it. I'll share a quick and easy way that works well for me.

First of all, θ is some unspecified angle. Our vector decomposition for $\vec{N}$ should work for any value of θ that we choose between 0° and 90°. What would happen if we chose $\theta = 0$? Then, the ramp would be horizontal, and the normal force would be vertical.

In Figure A.10a, I show the ramp, block, and normal force for a very small value of θ. Here, we see the ramp rotated slightly clockwise from horizontal, and the normal force rotated slightly clockwise from vertical. The angle by which the normal force is rotated from vertical is labeled ϕ in the figure.

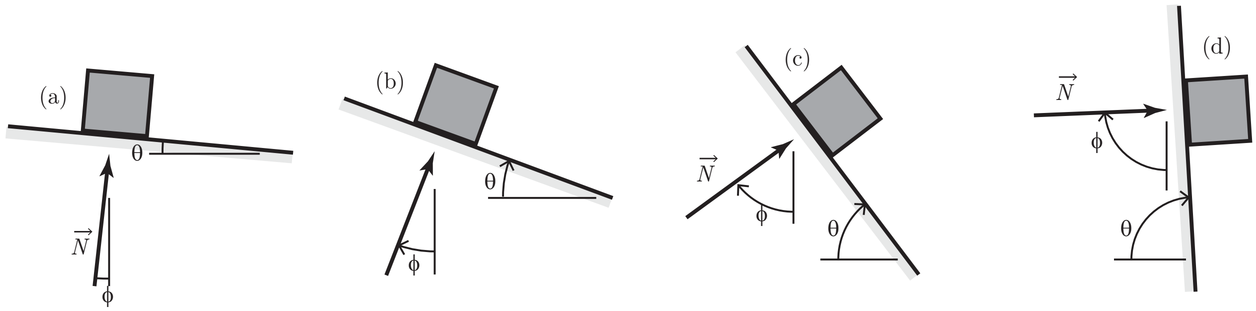


Figure A.10: Ramp, block, and normal force at different angles θ.

In parts (b), (c), and (d) of the figure, I show the system at different, larger, values of the ramp angle. Note that as the ramp rotates clockwise by the angle θ, the normal vector must rotate clockwise by the same angle. Otherwise the normal vector would not remain perpendicular to the ramp. Therefore, we know that $\phi = \theta$.

Notice that the angle ϕ, the angle of the normal vector relative to vertical, is the same as angle in the upper corner of the force triangle in Figure A.9. And since $\phi = \theta$, we can update the force triangle as shown in Figure A.11.

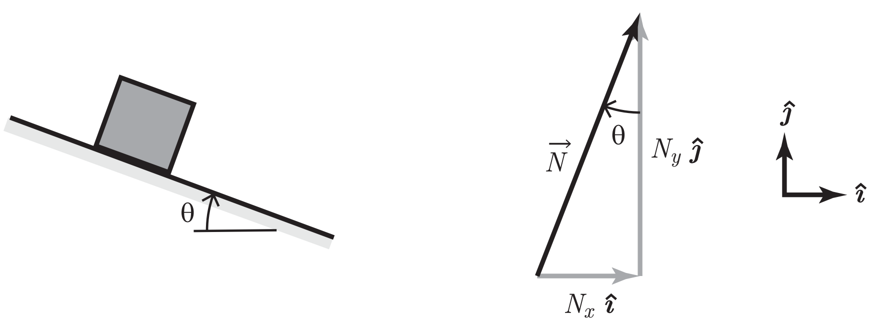


Figure A.11: Force triangle for the normal vector, with a key angle identified.

Step 3. Letting N denote the magnitude of the normal force $\vec{N}$, we are ready to calculate the magnitudes of the components:

$$|N_x| = N\,\sin(\theta), \qquad |N_y| = N\,\cos(\theta)$$

Step 4. Putting the component together, we get

$$\boxed{\vec{N} = N\,\sin(\theta)\,\hat{\imath} + N\,\cos(\theta)\,\hat{\jmath}.} \tag{A.7}$$

Note that both components are positive since the component vectors in Figure A.11 point in the same directions as the basis vectors $\hat{\imath}$ and $\hat{\jmath}$.

Step 5. Whenever you perform engineering analysis, or any type of analysis, it's a good idea to check that your answers make sense. For vector decomposition, one simple check we could perform is to test a few special cases where the answers are obvious.

For example, suppose we consider the case $\theta = 0$, when the ramp is completely horizontal. In this case, the normal vector should be upward and vertical: $\vec{N} = N\,\hat{\jmath}$. Incidentally, the friction force should point directly to the left: $\vec{F} = -F\,\hat{\imath}$.

Now, if we substitute $\theta = 0$ into Equations (A.7) and (A.6), we get $\cos(\theta) = 1$, $\sin(\theta) = 0$, and $\vec{N} = N\,\hat{\jmath}$, $\vec{F} = -F\,\hat{\imath}$. We find agreement.

Another case we can check easily is that when the ramp is vertical, $\theta = \pi/2 = 90°$. In this case the normal vector should point to the right ($\vec{N} = N\,\hat{\imath}$), and the friction vector should point straight up ($\vec{F} = F\,\hat{\jmath}$). Substituting $\cos(90°) = 0$ and $\sin(90°) = 1$ into (A.7) and (A.6), we find agreement with our expectations again.

A.4.4 A Different Set of Basis Vectors

Now let's decompose the same set of force vectors on the ramp/block problem, but this time let's use basis vectors $\hat{e}_t$ and $\hat{e}_n$ which are tangent and perpendicular (normal) to the ramp as shown in Figure A.12.

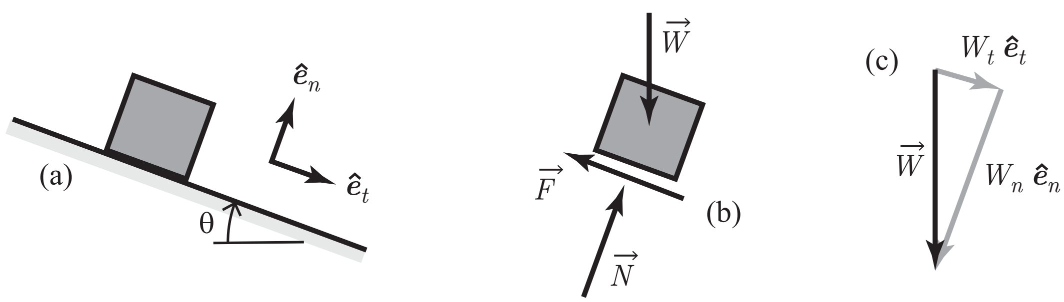


Figure A.12: The ramp/block problem again, but now we have basis vectors tangent and normal to the ramp.

Now, the friction force and normal force are easy to express in terms of the basis vectors:

$$\boxed{\vec{N} = N\,\hat{e}_n; \qquad \vec{F} = -F\,\hat{e}_t.} \tag{A.8}$$

The weight is the trickier vector. In Figure A.12c, we show the vector triangle in Step 1 of decomposing the weight. Observe that $\vec{W}$ is the hypotenuse, while the other two sides of the triangle are aligned with basis vectors $\hat{e}_t$ and $\hat{e}_n$.

In Step 2, we are to identify a key angle in the force triangle. Can you do it? Please try it before you begin reading the next paragraph.

If you did it correctly, you should have determined that bottom corner of the force triangle has angle θ. Given this, the decomposition of the weight vector is

$$\vec{W} = W\,\sin(\theta)\,\hat{e}_t - W\,\cos(\theta)\,\hat{e}_n,$$

or, upon equating the weight with mg,

$$\boxed{\vec{W} = mg\,\sin(\theta)\,\hat{e}_t - mg\,\cos(\theta)\,\hat{e}_n.}$$

I'll leave it to you to check that this decomposition makes sense.

A.4.5 Decomposition with Length Ratios

In the last decomposition example, we'll make one more modification. Rather than give you the ramp angle directly, suppose I tell you the lengths of two sides of the ramp. The lengths are labeled x and y in Figure A.13.

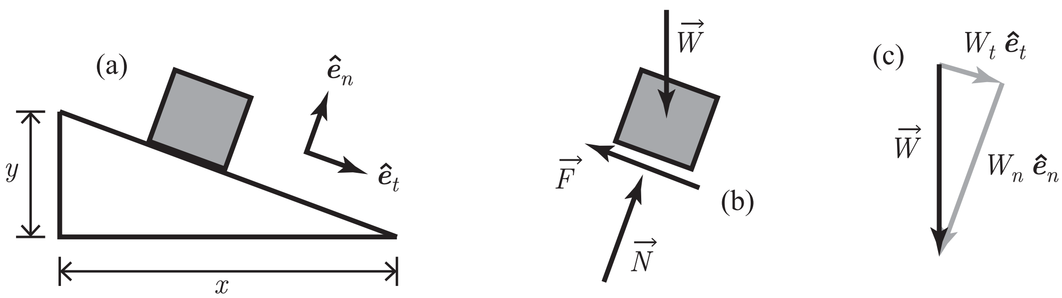


Figure A.13: The ramp/block problem in which ramp lengths are given.

The important thing to note is that the ramp triangle in Figure A.13a is *congruent* to the force vector triangle in Figure A.13c. Notice that I did not say the triangles are equal. They could not possibly be equal; the sides of one triangle is a set of lengths, whereas the sides of the other triangle are forces. But the interior angles of the triangles are the same. In addition, the ratios of the corresponding sides are the same. For example,

$$\frac{x}{y} = \frac{|W_n|}{|W_t|}.$$

Since the hypotenuse of the ramp triangle is $\sqrt{x^2+y^2}$, the following ratios are also equal:

$$\frac{y}{\sqrt{x^2+y^2}} = \frac{|W_t|}{mg}, \qquad \frac{x}{\sqrt{x^2+y^2}} = \frac{|W_n|}{mg}.$$

Notice that we can solve each of these quantities for the components $|W_t|$ and $|W_n|$:

$$|W_t| = mg\,\frac{y}{\sqrt{x^2+y^2}}, \qquad |W_n| = mg\,\frac{x}{\sqrt{x^2+y^2}}.$$

Therefore, we can write the decomposition as

$$\vec{W} = mg\,\frac{y}{\sqrt{x^2+y^2}}\,\hat{\boldsymbol{e}}_t - mg\,\frac{x}{\sqrt{x^2+y^2}}\,\hat{\boldsymbol{e}}_n. \tag{A.9}$$

That's your vector decomposition.

To check that our answer makes sense, let's observe that the ratio $x/\sqrt{x^2+y^2}$ is equal to the cosine (adjacent/hypotenuse) of the ramp angle. Similarly, $y/\sqrt{x^2+y^2}$ is equal to the sine (opposite/hypotenuse) of the ramp angle. Upon making this identification, you'll notice that (A.9) is equivalent to (A.4.4). Nice, eh?