ORIENTATION CONTROL NOTES

Compound Rotations & SLERPing

Let's define some orientations of a body

- When a body is in its reference orientation, this will correspond to $q = (1, \vec{0})$

- Let A denote the current orientation of the body, corresponding to quaternion q_A

  If $\vec{r}_0$ denotes the position of some arbitrary point on the body, relative to the body's origin, in the reference config, then

$$(0, \vec{r}_A) = q_A (0, \vec{r}_0)$$

or

$$r_A = q_A r_0 q_A^*$$

from p. 8 of 3D Rotations notes

Here $\vec{r}_A$ is the position of that point when the body is in the A orientation

- Let B denote a fixed target orientation, q_B, that you want the body to be in

  The position vector of that same point in the B orientation, we have

$$r_B = q_B r_0 q_B^*$$

If I desire to go directly to orientation B from orientation A, without going back to the reference config, then we can use just one rotation given by quaternion $q_{A\to B}$

So $r_B = q_B r_0 q_B^*$

$$r_B = q_{A\to B}\, r_A\, q_{A\to B}^*$$

$$r_B = q_{A\to B}\, q_A\, r_0\, q_A^*\, q_{A\to B}^*$$

This tells us that

$$q_B = q_{A\to B}\, q_A$$

so $\underbrace{q_{A\to B} = q_B q_A^*}$ Sometimes this is called the difference quaternion

This tells us how to get from A to B by rotating about a single axis by a specific angle. Such a rotation takes the body along the shortest path along a great circle on the 3-sphere in the 4-D quaternion coordinate space

The process of rotating from A to B along ~~the~~ difference quaternion $q_{A\to B}$ is called SLERPing. Here SLERP is the Spherical Linear intERPolation.

We write it as

$$\mathrm{SLERP}(q_A, q_B, h)$$

starting orientation

ending orient

parameter: $0 \to 1$

$$\mathrm{SLERP}(q_A, q_B, 0) = q_A$$

$$\mathrm{SLERP}(q_A, q_B, 1) = q_B$$

$q_{A\to B}$

$$q_B = (q_B q_A^*)\, q_A$$

The difference quaternion is a unit quaternion, so we can write it as

$$q_{A\to B} = \left(\cos(\theta_{AB}),\ \sin(\theta_{AB})\,\hat{v}_{AB}\right)$$

$\hat{v}_{AB}$ is the unit vector we rotate about

θ_{AB} is half the amount we rotate

Thus, we can write this continuous rotation from q_A to q_B along the great circle as

$$\mathrm{SLERP}(q_A, q_B, h) = \underbrace{\left(\cos(h\theta_{AB}), \sin(h\theta_{AB})\hat{v}_{AB}\right)}_{\substack{\text{This can be written compactly as } q_{A\to B}^{h} \\ \text{See exponentiation in QUATERNION PROPERTIES notes, p. 7,8}}} q_A$$

So if we're sitting at q_A & we want to go to q_B, which direction should we start moving (in q_0, q_1, q_2, q_3 space)?

Well,

$$\left.\frac{d}{dh}\right|_{h=0} \mathrm{slerp}(q_A, q_B, h) = \left.\theta_{AB}\left(-\sin(h\theta_{AB}), \cos(h\theta_{AB})\hat{v}_{AB}\right)\right|_{h=0} q_A$$

$$= \theta_{AB}\left(0, \hat{v}_{AB}\right) q_A$$

So, a unit vector in the direction of travel would be

$$\dot{q}_{slerp} = (0, \hat{v}_{AB})\, q_A \, ,$$

where $\hat{v}_{AB}$ is the unit axis vector for the difference $q_B q_A^*$

In general, $\dot{q}$ is NOT in the direction of $\dot{q}_{slerp}$.

$\dot{q}$ is related to the body's angular velocity $\omega_x, \omega_y, \omega_z$

$$\left.\begin{aligned}
\dot{q}_0 &= 0.5\left(-q_1\omega_x - q_2\omega_y - q_3\omega_z\right)\\
\dot{q}_1 &= 0.5\left(q_0\omega_x - q_3\omega_y + q_2\omega_z\right)\\
\dot{q}_3 &= 0.5\left(q_3\omega_x + q_0\omega_y - q_1\omega_z\right)\\
\dot{q}_4 &= 0.5\left(-q_2\omega_x + q_1\omega_y + q_0\omega_z\right)
\end{aligned}\right\} (\star)$$

We can write $\dot{q}$ as a sum of pieces tangent & perpendicular to $\dot{q}_{slerp}$

$$\dot{q} = \dot{q}_{\parallel} + \dot{q}_{\perp}$$

Note that $\dot{q}$ & $\dot{q}_{\parallel}$ are both tangent to the unit sphere in 4-D quaternion space.

$\Rightarrow \dot{q}_{\perp}$ is also tangent

Recall $\dot{q}_{slerp}$ is a unit quaternion

Note that $\dot{q}_{\parallel} = \dot{q} \cdot \dot{q}_{slerp}$

Therefore, we can write

$$\dot{q}_{\perp} = \dot{q} - \dot{q}_{\parallel}$$

$$\Rightarrow \dot{q}_{\perp} = \dot{q} - \dot{q} \cdot \dot{q}_{slerp}$$

So, what we'd like to do is

1.) Drive the perpendicular component of $\dot{q}$ to zero

2.) Drive the $\parallel$ component of $\dot{q}$ to the target valued, q_B