QUATERNION PROPERTIES

Define two quaternions

Scalar part

Vector part

$$q_A = S_A + x_A\hat{\imath} + y_A\hat{\jmath} + z_A\hat{k} = (S_A, \vec{V}_A)$$

$$q_B = S_B + x_B\hat{\imath} + y_B\hat{\jmath} + z_B\hat{k} = (S_B, \vec{V}_B)$$

How do the $\hat{\imath}, \hat{\jmath}, \hat{k}$ multiply

$$\hat{\imath}\hat{\imath} = \hat{\jmath}\hat{\jmath} = \hat{k}\hat{k} = -1$$

$$\hat{\imath}\hat{\jmath} = \hat{k} \qquad \hat{\jmath}\hat{\imath} = -\hat{k}$$

$$\hat{\jmath}\hat{k} = \hat{\imath} \qquad \hat{k}\hat{\jmath} = -\hat{\imath}$$

$$\hat{k}\hat{\imath} = \hat{\jmath} \qquad \hat{\imath}\hat{k} = -\hat{\jmath}$$

Now what is the product of q_A & q_B

$$q_A q_B = S_A S_B + (x_A\hat{\imath})(x_B\hat{\imath}) + (y_A\hat{\jmath})(y_B\hat{\jmath}) + (z_A\hat{k}\, z_B\hat{k})$$

$$+ S_A(x_B\hat{\imath} + y_B\hat{\jmath} + z_B\hat{k}) + S_B(x_A\hat{\imath} + y_A\hat{\jmath} + z_A\hat{k})$$

$$+ \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{bmatrix}$$

So

$$q_A q_B = \underbrace{S_A S_B - \vec{V}_A \cdot \vec{V}_B}_{\text{Scalar part}} + \underbrace{S_A\vec{V}_B + S_B\vec{V}_A + \vec{V}_A \times \vec{V}_B}_{\text{Vector part}}$$

Note that because of the cross product, quaternion multiplication is not commutative.

One can define a dot product for quaternions:

$$q_A \cdot q_B = s_A s_B + \vec{v}_A \cdot \vec{v}_B$$

$$= s_A s_B + x_A x_B + y_A y_B + z_A z_B$$

Note that $q \cdot q$ can be use to define a norm $\|q\| = \sqrt{q \cdot q} = \sqrt{s^2 + x^2 + y^2 + z^2}$

Conjugate:

Let $q = (s, \vec{v})$. Then $q^* = (s, -\vec{v})$

Some properties of conjugates:

1) $(q^*)^* = (s, -\vec{v})^* = (s, \vec{v}) = q$

So $(q^*)^* = q$

2) $(q_A q_B)^* = \left(s_A s_B - \vec{v}_A \cdot \vec{v}_B,\ -s_A \vec{v}_B - s_B \vec{v}_A - \vec{v}_A \times \vec{v}_B\right)$

Also

$$q_B^* q_A^* = \left(s_B s_A - (-\vec{v}_B)\cdot(-\vec{v}_A),\ s_B(-\vec{v}_A) + s_A(-\vec{v}_B) + (-\vec{v}_B)\times(-\vec{v}_A)\right)$$

$$= \left(s_B s_A - \vec{v}_B \cdot \vec{v}_A,\ -s_B \vec{v}_A - s_A \vec{v}_B + \underbrace{\vec{v}_B \times \vec{v}_A}_{-\vec{v}_A \times \vec{v}_B}\right)$$

So $\boxed{(q_A q_B)^* = q_B^* q_A^*}$

3) $(q_A + q_B)^* = q_A^* + q_B^*$ ← This one's obvious

4) $q q^* = \left(s^2 + \vec{v}\cdot\vec{v},\ \cancel{s\vec{v} + s(-\vec{v}) + \vec{v}\times(-\vec{v})}\right)$

$$= s^2 + \vec{v}\cdot\vec{v} = \|q\|^2$$

So $\boxed{q q^* = q^* q = \|q\|^2}$

5) This tells us that

$$q \frac{q^*}{\|q\|^2} = 1 \Rightarrow q^{-1} = \frac{q^*}{\|q\|^2}$$

Some more properties

6) $\|q^*\| = \sqrt{s^2 + (-\vec{v})\cdot(-\vec{v})} = \sqrt{s^2 + \vec{v}\cdot\vec{v}} = \|q\|$

$$\Rightarrow \boxed{\|q^*\| = \|q\|}$$

7) $\|q_A q_B\| = \sqrt{q_A q_B (q_A q_B)^*} = \sqrt{q_A q_B q_B^* q_A^*}$

$$= \sqrt{q_A \|q_B\|^2 q_A^*} = \sqrt{q_A q_A^* \|q_B\|^2}$$

$$= \sqrt{\|q_A\|^2 \|q_B\|} = \|q_A\| \|q_B\|$$

So $\boxed{\|q_A q_B\| = \|q_A\| \|q_B\|}$

Unit Quaternions

A unit quaternion is one for which $\|q\| = 1$.

Suppose $q = (s, \vec{v})$ is a unit quaternion,
Then there exists a unit vector $\hat{v}$ s.t.

$$\boxed{q = (\cos(\theta), \sin(\theta)\,\hat{v})}$$

Proof

- If $q = (1, \vec{0})$, then $\theta = 0$, $\hat{v}$ is arbitrary
- If $q \neq (1, \vec{0})$, let $K = |\vec{v}|$, then define $\hat{v} = \frac{1}{K}\vec{v}$

$$\Rightarrow q = (s, K\hat{v})$$

$$\|q\| = s^2 + K^2(\hat{v}\cdot\hat{v}) = \underbrace{s^2 + K^2 = 1}_{\text{Eqn for a circle of radius 1}}$$

There exists a $\theta \in (-\pi, \pi)$ s.t.

$$s = \cos(\theta) \qquad K = \sin(\theta)$$

$$\Rightarrow q = (s, \vec{v}) = (s, K\hat{v}) = (\cos(\theta), \sin(\theta)\,\hat{v}) \qquad \boxtimes$$

More properties of unit quaternions

1) if q_A & q_B are unit quaternions, then then the product $q_A q_B$ is a unit quaternion

This is because

$$\underbrace{\|q_A q_B\| = \|q_A\|\,\|q_B\|}_{\text{from property 7 on p. 4}} = 1 \cdot 1 = 1$$

2) If q is a unit quaternion, then

$$q^{-1} = \frac{q^*}{\|q\|^2} = q^* \quad , \text{ since } \|q\| = 1$$

Property (5) on p. 5

$$\boxed{q^{-1} = q^*}$$

Property 6, p. 4

3) $\|q^{-1}\| = \|q^*\| = \|q\| = 1$

So $\|q^{-1}\|$ is a unit quaternion if q is.

Log Function

Let q be a unit quaternion, so I can write

$$q = (\cos(\theta), \sin(\theta)\hat{v})$$

Then the log function is defined as

$$\boxed{\log(q) = (0, \theta\hat{v})}$$

Note that

$$\log((1, \vec{0})) = (0, \vec{0})$$

just like real numbers

Also note that $\log(q)$ generally not a unit vector

Exponential Function

Suppose we have a (pure) quaternion of the form $q = (0, \theta \hat{v})$

unit vector

Real #

Then we define

$$\exp(q) = (\cos(\theta), \sin(\theta)\hat{v})$$

So, the log & exponential are mutually inverse.

Exponentiation

Let q be a unit quaternion, then for $t \in \mathbb{R}$ Exponentiation q^t is defined as

$$\boxed{q^t := \exp(t \log(q))}$$

Lets write this out...

$$q^t = (\cos(\theta), \sin(\theta)\hat{v})^t = \exp(t(0, \theta\hat{v}))$$

$$= \exp(0, t\theta\hat{v}) = (\cos(t\theta), \sin(t\theta)\hat{v})$$

So, another way we can write exponentiation is

$$\boxed{q^t = \left(\cos(\theta t),\ \sin(\theta t)\hat{v}\right)}$$

Products of Exponentials

Suppose $\underset{\mathbb{R}}{a}\underset{\text{quat}}{V} = (0, a\vec{v})$ & $\underset{\mathbb{R}}{b}\underset{\text{quat}}{V} = (0, b\vec{v})$

what is the product

$$\exp(aV)\exp(bV)\ ?$$

$$\Rightarrow \left(\cos(a),\ \sin(a)\vec{v}\right)\left(\cos(b),\ \sin(b)\vec{v}\right)$$

$$= \Big(\cos(a)\cos(b) - \sin(a)\sin(b)(\vec{v}\cdot\vec{v}),$$
$$+\left[\cos(a)\sin(b) + \cos(b)\sin(a)\right]\vec{v} + \sin(a)\sin(b)(\vec{v}\times\vec{v})\Big)$$

$$= \left(\cos(a+b),\ \sin(a+b)\vec{v}\right) = \exp((a+b)V)$$

So: $$\boxed{\exp(aV)\exp(bV) = \exp((a+b)V)}$$