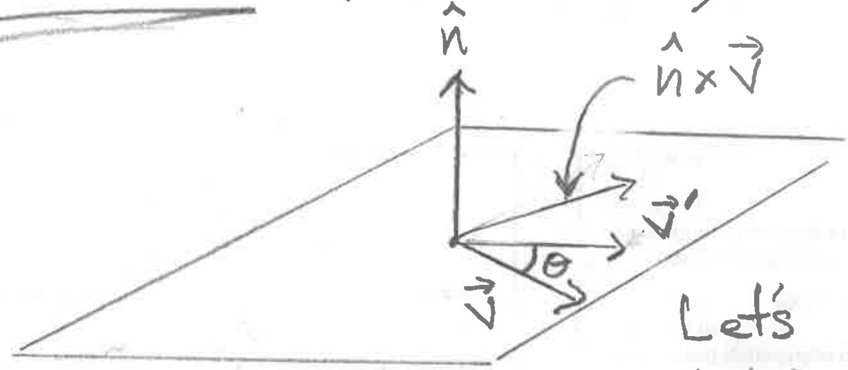
3D ROTATIONS

3D Rotations (Special Case)



Suppose we have unit vector $\hat{n}$, and a vector $\vec{v}$ s.t. $\vec{v}\perp\hat{n}$

Let's find vector $\vec{v}'$ obtained by rotating $\vec{v}$ around $\hat{n}$ by angle θ

Define $\hat{n}\times\vec{v}$. Note that $\hat{n}\times\vec{v}$ lies in same plane as $\vec{v}$ & $\vec{v}'$

Note thate $\vec{v}$, $\vec{v}'$, & $\hat{n}\times\vec{v}$ all have same magnitude.

Then, we can write

$$\boxed{\vec{v}' = \cos(\theta)\,\vec{v} + \sin(\theta)(\hat{n}\times\vec{v})} \quad (\#)$$

Let's write this in terms of quaternions

Start by writing the vectors $\vec{v}$, $\vec{v}'$, & $\hat{n}$ as quaternions with zero scalar part

$$v = (0, \vec{v}), \quad v' = (0, \vec{v}'), \quad n = (0, \hat{n})$$

Observe that quaternion nv is

$$nv = (\cancel{-\hat{n}\cdot\vec{v}}^{\,0}, \hat{n}\times\hat{v}) = (0, \hat{n}\times\hat{v})$$

So, I can write (#) on p.1 in quaternions as

$$v' = \cos(\theta)\, v + \sin(\theta)\, n v$$

or

$$v' = \left(\cos(\theta) + \sin(\theta)\, n\right) v$$

$$\Rightarrow v' = \underbrace{\left(\cos(\theta), \sin(\theta)\,\hat{n}\right) v}_{\text{Let's verify this}} \qquad (*)$$

$$\left(\cos(\theta), \sin(\theta)\,\hat{n}\right)(0, \vec{v})$$

$$= \left(\sin(\theta)\,\cancelto{0}{\hat{n}\cdot\vec{v}}, \cos(\theta)\,\vec{v} + \sin(\theta)\,\hat{n}\times\vec{v}\right)$$

$$= \left(0, \cos(\theta)\,\vec{v} + \sin(\theta)\,\hat{n}\times\vec{v}\right) = \vec{v}' \checkmark$$

Observe that the quaternion

$$\left(\cos(\theta), \sin(\theta)\,\hat{n}\right) \text{ in } (*)$$

is a unit quaternion & can be written as

$$\exp(\theta \hat{n})$$

So

$$\boxed{v' = \exp(\theta \hat{n})\, v}$$

One more observation.

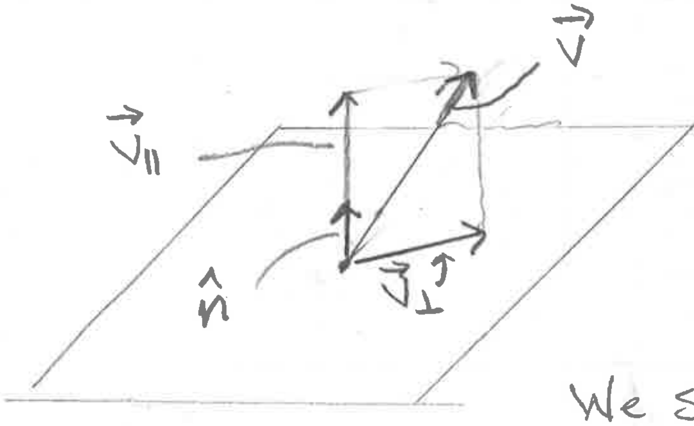
$$\exp(-\theta n) = (\cos(-\theta), \sin(-\theta)\hat{n})$$
$$= (\cos(\theta), -\sin(\theta)\hat{n})$$
$$= (\exp(\theta\hat{n}))^*$$

Also

$$v\exp(-\theta n) = (0, \vec{v})(\cos(\theta), -\sin(\theta)\hat{n})$$
$$= (0, \cos(\theta)\vec{v} - \sin(\theta)\,\vec{v}\times\hat{n})$$
$$= (0, \cos(\theta)\vec{v} + \sin(\theta)\,\hat{n}\times\hat{v})$$

So $\boxed{v' = v\exp(-\theta n)}$ also

3D Rotations, General Case
Rodriguez Formula



Now, we'll consider the case where $\vec{v}$ is **NOT** $\perp$ to $\hat{n}$.

We want to rotate $\vec{v}$ about $\hat{n}$ by angle θ.

We start by splitting $\vec{v}$ into two pieces

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$$

Here, $\vec{v}_{\parallel}$ is parallel to $\hat{n}$
$\vec{v}_{\perp}$ is perpendicular to $\hat{n}$

The rotated vector is

$$\vec{v}\,' = \vec{v}_{\parallel}' + \vec{v}_{\perp}'$$

$\vec{v}_{\parallel}' = \vec{v}_{\parallel}$ since rotation of the parallel component has no effect

$$v_{\perp}' = \cos(\theta)\,\vec{v}_{\perp} + \sin(\theta)\left(\hat{n}\times\vec{v}_{\perp}\right),$$

as before

So

$$\vec{v}\,' = \vec{v}_{\parallel} + \cos(\theta)\,\vec{v}_{\perp} + \sin(\theta)\left(\hat{n}\times\vec{v}_{\perp}\right) \qquad (\blacktriangle)$$

Observe that

$$\vec{v}_\perp = \vec{v} - \vec{v}_\parallel$$

Also $\hat{n}\times\vec{v} = \cancelto{0}{(\hat{n}\times\vec{v}_\parallel)} + \hat{n}\times\vec{v}_\perp$

$$\hat{n}\times\vec{v} = \hat{n}\times\vec{v}_\perp$$

So (▲) on previous page can be written as

$$\vec{v}\,' = \vec{v}_\parallel - \cos(\theta)\vec{v}_\parallel + \cos(\theta)\vec{v} + \sin(\theta)\left(\hat{n}\times\vec{v}\right)$$

Furthermore

$$\vec{v}_\parallel = (\vec{v}\cdot\hat{n})\,\hat{n}$$

So

$$\boxed{\vec{v}\,' = \left(1-\cos(\theta)\right)(\vec{v}\cdot\hat{n})\,\hat{n} + \cos(\theta)\vec{v} + \sin(\theta)\left(\hat{n}\times\vec{v}\right)}$$

General 3D Rotations with Quaternions

Using previous construction, lets define

$n = (0, \hat{n})$, $v_{\parallel} = (0, \vec{v}_{\parallel})$, $v_{\perp} = (0, \vec{v}_{\perp})$, $v = (0, \vec{v})$

$v'_{\perp} = (0, \vec{v}'_{\perp})$, $v' = (0, \vec{v}')$

We found previously that

$$v' = v_{\parallel} + v'_{\perp}$$

But, we found on p.2 that

$$v'_{\perp} = \exp(\theta n) v_{\perp}$$

So $$v' = \underbrace{v_{\parallel}} + \underbrace{\exp(\theta n) v_{\perp}} \qquad (\blacksquare)$$

Let's look at each of these pieces separately, starting with the second.

Note $$\exp(\theta n) v_{\perp} = \exp(\tfrac{\theta}{2} n) \underbrace{\exp(\tfrac{\theta}{2} n) v_{\perp}}$$

On p.3 we showed that we can reverse the order, with a conjugate

So $$\exp(\theta n) v_{\perp} = \exp(\tfrac{\theta}{2} n) v_{\perp} \exp(-\tfrac{\theta}{2} n) \qquad (\bigstar)$$

Now, let's look at the first term of (■)

$$V_{\parallel} = \underbrace{\exp\left(\tfrac{\theta}{2}n\right) \exp\left(\tfrac{-\theta}{2}n\right)}_{\text{Identity } (1,\vec{0})} V_{\parallel}$$

Since $\vec{V}_{\parallel}$ is parallel to $\hat{n}$, these two pieces commute

Check it

$$\exp\left(-\tfrac{\theta}{2}n\right)V_{\parallel} = \left(\cos\left(\tfrac{\theta}{2}\right), -\sin\left(\tfrac{\theta}{2}\right)\hat{n}\right)\left(0, \vec{V}_{\parallel}\right)$$

$$= \left(-\sin\left(\tfrac{\theta}{2}\right)\hat{n}\cdot\vec{V}_{\parallel},\ \cos\left(\tfrac{\theta}{2}\right)\vec{V}_{\parallel} - \sin\left(\tfrac{\theta}{2}\right)\cancelto{0}{\hat{n}\times\vec{V}_{\parallel}}\right)$$

since $\hat{n}$ & $\vec{V}_{\parallel}$ are parallel

$$V_{\parallel}\exp\left(\tfrac{-\theta}{2}n\right) = \left(0, \vec{V}_{\parallel}\right)\left(\cos\left(\tfrac{\theta}{2}\right), -\sin\left(\tfrac{\theta}{2}\right)\hat{n}\right)$$

$$= \left(-\sin\left(\tfrac{\theta}{2}\right)\hat{n}\cdot\vec{V}_{\parallel},\ \cos\left(\tfrac{\theta}{2}\right)\vec{V}_{\parallel} - \sin\left(\tfrac{\theta}{2}\right)\cancelto{0}{\left(\vec{V}_{\parallel}\times\hat{n}\right)}\right)$$

Both are the same!

So

$$V_{\parallel} = \exp\left(\tfrac{\theta}{2}n\right) V_{\parallel} \exp\left(\tfrac{-\theta}{2}n\right) \quad (\infty)$$

Substituting (∞) & (★) into (▤), we get

$$V' = \exp\left(\tfrac{\theta}{2}n\right) V_{\parallel} \exp\left(\tfrac{-\theta}{2}n\right) + \exp\left(\tfrac{\theta}{2}n\right) V_{\perp} \exp\left(\tfrac{-\theta}{2}n\right)$$

Combining terms : $V_{\parallel} + V_{\perp} = V$

$$V' = \exp\left(\tfrac{\theta}{2}n\right) V \exp\left(\tfrac{-\theta}{2}n\right)$$

This is a general V. There is no stipulation about being $\parallel$ or $\perp$ to $\hat{n}$

This is a general unit quaternion. Let's call it q.

Then $\exp\left(\tfrac{-\theta}{2}n\right) = q^* = q^{-1}$

So, we can write

$$\boxed{V' = q V q^*}$$

Here's your general rotation